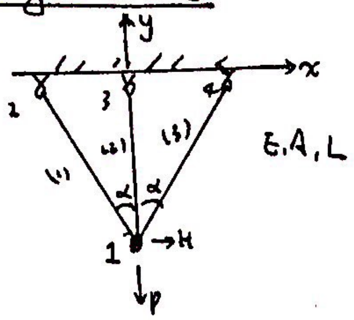
CSMD Assignment 1.1 Shushu Qin.



1. Master Stiffness Equations.

Element stiffness matrix can be calculated as 

$$\underset{\sim}{k}^e = \frac{E^eA^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$ in which $c=\cos\varphi$ $s=\sin\varphi$

$\underset{\sim}{k}^e \vec{u}^e = \vec{f}^e$

Element (1) $\varphi = \alpha + \frac{\pi}{2}$, $L^e = L/\cos\alpha$.

$$\underset{\sim}{k}^{(1)} = \frac{EA}{L/c}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}\begin{matrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{matrix}$$ in which $c=\cos\alpha$, $S=\sin\alpha$.

Similarly

$$\underset{\sim}{k}^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{matrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{matrix}$$

$$\underset{\sim}{k}^{(3)} = \frac{EA}{L/c}\begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix}\begin{matrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{matrix}$$

Then we assemble the element matrix to obtain the global one, which gives us $\underset{\sim}{k}\vec{u} = \vec{f}$

$\underset{\sim}{k} = \frac{EA}{L}$

s^2c $+s^2c$	$-sc^2$ $+sc^2$	$-s^2c$	sc^2			$-s^2c$	$-sc^2$
$-sc^2$ $+sc^2$	c^3+1 $+c^3$	sc^2	$-c^3$		-1	$-sc^2$	$-c^3$
$-s^2c$	sc^2	s^2c	$-sc^2$				
sc^2	$-c^3$	$-sc^2$	c^3				
	-1				1		
$-s^2c$	$-sc^2$					s^2c	sc^2
$-sc^2$	$-c^3$					sc^2	c^3.

(1) •
(2) •
(3) •

$$= \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ \cdot & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3. \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & \text{symm} & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}$$

The known applied forces are $f_{x1}=H$ $f_{y1}=-P$. Th[illegible]
[illegible] system.

Therefore we have $\underset{\sim}{K}\begin{bmatrix} u_{x1}\\ u_{y1}\\ u_{x2}\\ u_{y2}\\ u_{x3}\\ u_{y3}\\ u_{x4}\\ u_{y4}\end{bmatrix}=\begin{bmatrix} H\\ -P\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$

The 5th row contains only zeros because the internal forces of bar (2) is always along the bar. Hence $f_{x3}\equiv 0$ no matter the external forces are, which requires all the stiffness terms at the 5th row to be zero.

The 5th column contains only zeros because the displacement of node 3 in x direction won't generate force in any of the bars.

2. Apply displacement BCs $u_{x2}=u_{y2}=u_{x3}=u_{y3}=u_{x4}=u_{y4}=0$

The system is reduced to

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0\\ 0 & 1+2c^3\end{bmatrix}\begin{bmatrix} u_{x1}\\ u_{y1}\end{bmatrix}=\begin{bmatrix} H\\ -P\end{bmatrix}$$

3. Solve for u_{x1} and u_{y1} $\begin{cases} u_{x1}=\frac{L}{EA}\frac{H}{2cs^2}\\ u_{y1}=-\frac{L}{EA}\frac{P}{1+2c^3}\end{cases}$

When $\alpha\to 0$, the three bars are almost of the same configuration 
so they equally share the force in y direction, $u_{y1}=-\frac{L}{EA}\frac{P}{3}$

It can hardly resist force in x direction, so u_{x1} "blow up" if $H\neq 0$.

When $\alpha\to\pi/2$, bar (1) and (2) are almost horizontal and infinitely long. They can not resist vertical forces, so $u_{y1}=-\frac{L}{EA}P$

4. Recover all the axial forces.

Element (1)

Convert to local displacements using $\bar{\vec{u}}^{(1)}=\underset{\sim}{T}^{(1)}\vec{u}^{(1)}$

$$\bar{\vec{u}}^{(1)}=\begin{bmatrix} -s & c & & \\ -c & -s & & \\ & & -s & c\\ & & -c & -s\end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2}\\ -\frac{L}{EA}\frac{P}{1+2c^3}\\ 0\\ 0\end{bmatrix}=\frac{L}{EA}\begin{bmatrix} -\frac{H}{2sc}-\frac{Pc}{1+2c^3}\\ -\frac{H}{2s^2}+\frac{Ps}{1+2c^3}\\ 0\\ 0\end{bmatrix}$$

$$F^{(1)}=\frac{EA}{L/c}(\bar{u}_{x2}-\bar{u}_{x1})=\frac{H}{2s}+\frac{Pc^2}{1+2c^3}$$

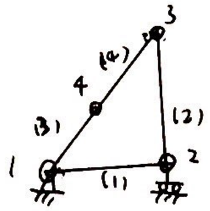
Similarly

$$\vec{u}^{(2)}=-\frac{L}{EA}\begin{bmatrix} \frac{P}{1+2c^3}\\ -\frac{H}{2cs^2}\\ 0\\ 0\end{bmatrix}\qquad F^{(2)}=\frac{P}{1+2c^3}$$

$$\vec{u}^{(3)} = \frac{L}{EA}\begin{bmatrix} \frac{H}{2SC} - \frac{PC}{1+2C^3} \\ \frac{H}{2S^2} + \frac{PS}{1+2C^3} \\ 0 \\ 0 \end{bmatrix} \qquad F^{(3)} = -\frac{H}{2S} + \frac{PC^2}{1+2C^3}$$

If $\alpha \to 0$, $H \neq 0$, $F^{(3)} \to \infty$. This can be explained by the same reason for u_{x1} "blow up" in 3.

Assignment 1.2



Element (1) and (2) have the same stiffness matrix as shown in the slides.

$$\vec{f}^{(1)} = 10\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

$$\vec{f}^{(2)} = 5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Element (3)

$$\vec{f}^{(3)} = 20\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Element (4)

$$\vec{f}^{(4)} = 20\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Assemble the matrices.

$$\underset{\sim}{K} = \begin{bmatrix} 10+20 & 20 & -10 & 0 & & & -20 & -20 \\ 20 & 20 & 0 & 0 & & & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 5 & 0 & -5 & & \\ & & 0 & 0 & 20 & 20 & -20 & -20 \\ & & 0 & -5 & 20 & 5+20 & -20 & -20 \\ -20 & -20 & & & -20 & -20 & 20+20 & 20+20 \\ -20 & -20 & & & -20 & -20 & 20+20 & 20+20 \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & \text{sym} & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix}$$

Apply Dirichlet boundary conditions. the reduced system is

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ & 20 & 20 & -20 & -20 \\ & & 25 & -20 & -20 \\ & \text{sym} & & 40 & 40 \\ & & & & 40 \end{bmatrix}\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \vec{f}$$

The stiffness matrix is singular as Row 4 and Row 5 are the same.

Physically it makes sense because the structure becomes a mechanism. Node 4 can move freely as long as $u_{x4} = -u_{y4}$.