<u>Assignment 1.1</u>

On "The Direct Stiffness Method"

Consider the truss problem defined in the figure 1.1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

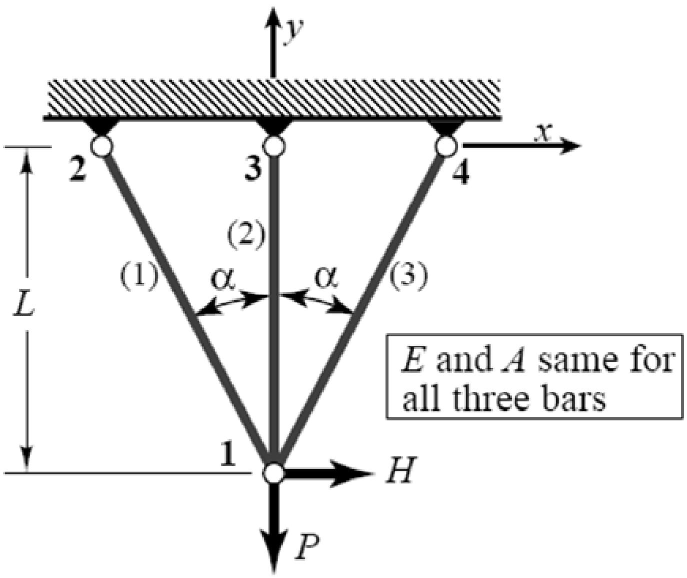

Figure 1.1.- Truss structure. Geometry and mechanical features

1. Show that the master stiffness equations are,

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which c = cosα and s = sinα. Explain from physics why the 5th row and column contain only zeros.

2. Apply the BC's and show the 2-equation modified stiffness system.

3. Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

4. Recover the axial forces in the three members. Partial answer: $F^{(3)} = -H/(2s) + Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

5. Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

Assignment 1.2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

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The assignment must be submitted as a pdf file named **As1-Surname.pdf** to the CIMNE virtual center.

CSMD: Assignment 1 and 2

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1 Assignment 1

1.1 Master stiffness equations

Consider the general expression for the elemental stiffness matrix and elemental force vector for a pin-jointed element α degrees inclined with respect to the horizontal axis:

$$\mathbf{K}^e = \mathbf{C}^T \tilde{\mathbf{K}}^e \mathbf{C} \tag{1}$$

Where $\tilde{\mathbf{K}}^e$ and $\mathbf{C}$ are the elemental stiffness in local axes and Rotation matrix respectively:

$$\tilde{\mathbf{K}}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \ \mathbf{R} = \begin{bmatrix} c_\phi & s_\phi & 0 & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ 0 & 0 & c_\phi & s_\phi \\ 0 & 0 & -s_\phi & c_\phi \end{bmatrix} \tag{2}$$

with $c_\phi = \cos(\phi_e)$ and $s_\phi = \sin(\phi_e)$, where ϕ_e is the angle with respect to the horizontal axis of element e.

The result of (1) yields the following stiffness matrix for each element:

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c_\phi^2 & s_\phi c_\phi & -c_\phi^2 & -s_\phi c_\phi \\ s_\phi c_\phi & s_\phi^2 & -s_\phi c_\phi & -s_\phi^2 \\ -c_\phi^2 & -s_\phi c_\phi & c_\phi^2 & s_\phi c_\phi \\ -s_\phi c_\phi & -s_\phi^2 & s_\phi c_\phi & s_\phi^2 \end{bmatrix} \tag{3}$$

Now, for computing the elemental stiffness matrices, we have to take into account the values of A^e, E^e, L^e and ϕ_e. The area A and Young's modulues E are constant for every element. Element 2 length is already given by the initial data $L^2 = L$, and as every superior node is located at the same horizontal line, L^1 and L^3 can be expressed as $L^1 = L^3 = \frac{L}{cos(\alpha)}$. Moreover, it can be seen that the different angles ϕ_e can be also particularized as

$$\phi_1 = \frac{\pi}{2} + \alpha; \quad \phi_2 = \frac{\pi}{2}; \quad \phi_3 = \frac{\pi}{2} - \alpha, \tag{4}$$

and then:

$$
\begin{aligned}
&sin(\phi_1) = cos(\alpha); && cos(\phi_1) = -sin(\alpha); \\
&cos(\phi_2) = 0; && sin(\phi_2) = 1; \\
&sin(\phi_3) = cos(\alpha); && cos(\phi_3) = sin(\alpha)
\end{aligned}
\tag{5}
$$

Substituting for these expressions and for the geometrical and material properties of each element into (3), we get the elemental matrices in the general axis:

$$
\mathbf{K}^1 = \frac{cEA}{L}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}
$$

$$
\mathbf{K}^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\tag{6}
$$

$$
\mathbf{K}^3 = \frac{cEA}{L}\begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix}
$$

Assembling the matrices according to the jointed nodes, we can see that the general stiffness matrix is

$$
\mathbf{K} = \frac{EA}{L}\begin{bmatrix}
2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\
 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\
 & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\
 & & & c^3 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & symm & & & 1 & 0 & 0 \\
 & & & & & & cs^2 & c^2s \\
 & & & & & & & c^3
\end{bmatrix}
\tag{7}
$$

And because the only external forces applied are done over node 1, the system of equation becomes (positive forces if they have the positive direction given by the axis), the system of equations given by the Direct Stiffness Method is:

$$
\mathbf{Ku} = \mathbf{F} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & symm & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The fifth row of the general stiffness matrix is full of zeros because it makes reference to the horizontal actions acting on bar 2. As the bar is vertical, these forces would create a bending moment, which cannot be possible given the fact that the bar is pin-jointed (only axial forces allowed as internal forces).

1.2 Modified system

BC are null vertical and horizontal displacements at nodes 2,3 and 4 ($\mathrm{u_{x2}} = \mathrm{u_{y2}} = \mathrm{u_{x3}} = \mathrm{u_{y3}} = \mathrm{u_{x4}} = \mathrm{u_{y4}} =$). The resulting system is a 2x2 matricial system:

$$
\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{8}
$$

1.3 Solve for u_{x1} and u_{y1}

Inverting (8) we find the values of u_{x1} and u_{y1}

$$
\begin{cases} u_{x1} = \dfrac{H\ L}{EA2cs^2} \\ u_{y1} = \dfrac{-P\ L}{EA(1+2c^3)} \end{cases} \tag{9}
$$

For the limit case of $\alpha \to \frac{\pi}{2}$, $c \to 0$ and $s \to 1$ and so, the strain is almost null ($\frac{ux1}{L} \to 0$), meaning that the horizontal displacement increases as the lengths (or angles α)of bar 1 and 2 increase.

For the limit case of $\alpha \to 0$, u_{x1} "blows up" for $H \neq 0$ because the structure becomes a system of vertical bars in the same position. As one of the two forces applied is horizontal and all the bars are pin-jointed, we are inducing a rotation in the system that does not cause any bending moment, so the system can rotate freely whatever the value of H.

1.4 Axial forces

Considering the equilibrium of forces at node 1 and the partial solution already given the following system is obtained:

$$\begin{cases} F_1 s - F_3 s = H \\ F_1 c + F_2 + F_3 c = P \\ F_3 = \dfrac{-H}{2s} + \dfrac{Pc^2}{1+2c^3} \end{cases} \tag{10}$$

With positive forces if the bar is experiencing tractions.

We can directly substitute F_3 in the first equation to obtain $F_1 = \frac{H}{2s} + P\frac{c^2}{1+2c^3}$ and then obtain $F_2 = P - c\,(F_1 + F_3) = P - c\frac{Pc^2}{2(1+2c^3)}$.

We can see that for the limit case $\alpha \to 0$ axial forces F_3 and F_1 become infinite unless H = 0. The reason is the same explained before: our model does not handle bending moments.

2 Assignment 2

2.1 Include 1 more node. Explain solution

New bar 3 stiffness is two times higher than the original stiffness matrix for bar 3, as all directions and parameters remain the same except for the length, which has halved. New bar 4 stiffness matrix is the opposite, as the bar has the same characteristics but the orientation.

The stiffness matrices are

$$\begin{aligned} \mathbf{K}^1 &= \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathbf{K}^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix} \\ \mathbf{K}^3 = -\mathbf{K}^4 &= \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix} \end{aligned} \tag{11}$$

Assembling them into **K** we find the new general stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & symm & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \tag{12}$$

Again, applying BC of null vertical displacements at nodes 1 and 2, and no horizontal displacements at node 1, we eliminate rows and columns 1,2 and 4. The following system is obtained, once the same process has been applied to displacements and forces vectors:

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ & 20 & 20 & -20 & -20 \\ symm & & 25 & -20 & -20 \\ & & & 40 & 40 \\ & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ f_{x3} \\ f_{y3} \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Mathematically speaking, the system is singular because its rank is lower than its dimension (rows 4th and 5th are equal). Physically speaking, the system is singular because we are creating a mechanism of 4 pin-jointed bars (see Figure 1). There are infinite configurations of this quadrilateral and thus, infinite solutions to the problem.

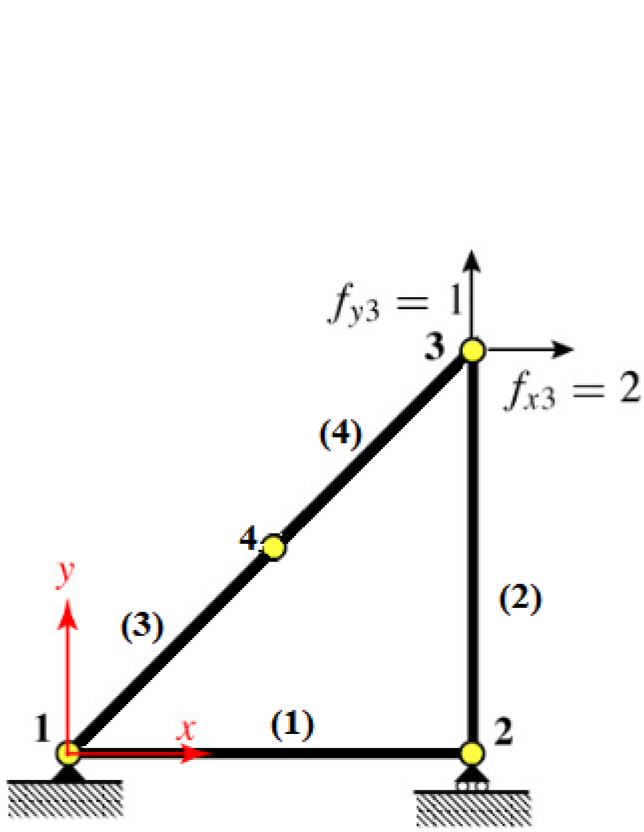

Figure 1: Scheme of the modified structure for Assignment 2. Element numbering between brackets