MASTER'S DEGREE NUMERICAL METHODS IN ENGINEERING

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Homework 10: Dynamic

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In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

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A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

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1 Assignment 10.1

In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

The dynamic system is shown in Figure 1, where the applied force $r(t) = F$ is a constant.

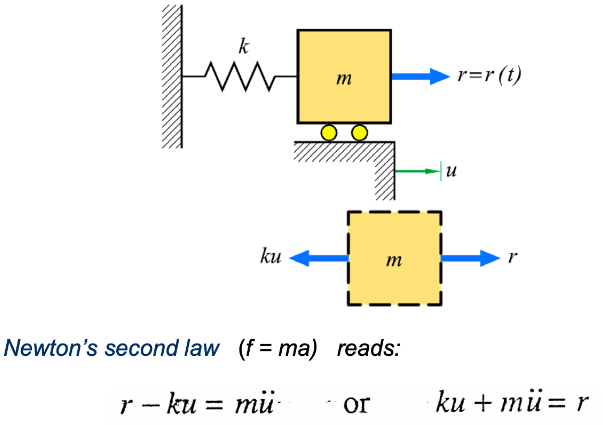

Figure 1: Dynamic system.

For an harmonic system the solution will be in the shape of Eq. (1) in order to have a initial displacement in the shape of F/k and a initial velocity $\dot{u}\ (t = 0) = 0$.

$$u(t) = F/k - F/k \cdot cos(\omega t) \tag{1}$$

$$\dot{u}(t = 0) = +F/k \cdot \omega \cdot sin(\omega 0) = 0$$

$$\ddot{u}(t) = +F/k \cdot \omega^2 \cdot cos(\omega t)$$

Using Newton's second law reads as Eq. (2):

$$[k\ \ (F/k - F/k \cdot cos(\omega t)) + m \cdot F/k \cdot \omega^2 \cdot cos(\omega t)] = F \tag{2}$$

$$-F \cdot cos(\omega t) + m \cdot F/k \cdot \omega^2 \cdot cos(\omega t) = 0$$

$$-1 + m/k \cdot \omega^2 = 0$$

$$\omega = \pm\sqrt{k/m}$$

2 Assignment 10.2

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

Considering from the previous exercise that the frequency is related to the ratio between the stiffness k and the mass m ($\omega = \sqrt{k/m}$) it is important first to determine k. To this end, the force applied to perform a displacement of the bar will be calculated.

The only force generating a displacement is the mass placed at the middle of the axial bar, doing a force in the direction of the gravity, and the maximum displacement of a punctual load in the middle length of a clamped bar is:

$$\delta_{max} = \frac{F \cdot L^3}{192 \cdot EJ}$$

where F is the force generated by the mass, L is the length of the bar, E is the Young's Modulus and J is the inertia of the bar in the direction of interest. These constants will be determined, assuming the bar is a square:

$$J = b \cdot h^3/12 = A^2/12$$

$$F = m \cdot g$$

$$\delta_{max} = \frac{m\ g\ L^3}{192 \cdot E \cdot A^2/12} = \frac{m\ g\ L^3}{16 \cdot E \cdot A^2}$$

Then the stiffness k is defined as the force to apply to generate a unite movement. Then considering that the force is $F = m \cdot g$, the stiffness reads:

$$k = \frac{F}{\delta_{max}} = \frac{m \cdot g}{\frac{m\ g\ L^3}{16 \cdot E \cdot A^2}} = \frac{16 \cdot E \cdot A^2}{L^3}$$

Then, it is possible to calculate the frequency of the structure:

$$\omega = \pm\sqrt{\frac{16\ E\ A^2}{m\ L^3}}$$

3 Assignment 10.3

Use the expression on slide 18 to derive the mass matrix of slide 17.

To obtain the expression for the *consistent mass matrix* for 1D linear elements, the expression shown in *slide 17* written in Eq. (3), requires to use the shape functions. For the sake of simplicity, isoparametric functions are chosen, and the important equations are presented:

$$m = \int N^T\ N\ \rho\ dV \tag{3}$$

$$N_1(\xi) = \frac{1}{2} \cdot (1 - \xi)$$

$$N_2(\xi) = \frac{1}{2} \cdot (1 + \xi)$$

Then the matrix form of the expression of Eq. (3) is:

$$m = \int_{-1}^{1} \frac{1}{2} \begin{bmatrix} (1-\xi) \\ (1+\xi) \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} (1-\xi) & (1+\xi) \end{bmatrix}\ \rho\ A\ |J|\ d\xi$$

where the Jacobian of the transformation is constant and equal to $J = L/2$:

$$m = \frac{1}{8}\rho\ A\ L \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi^2) \\ (1-\xi^2) & (1+\xi)^2 \end{bmatrix}\ d\xi = \frac{1}{8}\rho\ A\ L \begin{bmatrix} 8/3 & 4/3 \\ 4/3 & 8/3 \end{bmatrix}$$

The final consistent mass matrix is:

$$m = \rho\ A\ L \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

4 Assignment 10.4

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2.

Now the difference will be that A is not a constant anymore, therefore it will not be possible to take it out from the integral. The equation used to described $A(x)$ is shown in Eq. (4).

$$A(x) = A_1 + (A_2 - A_1) \cdot \frac{x}{L} \tag{4}$$

Using isoparametric description for $A(x)$

$$A(\xi) = \frac{A_1}{2} \cdot (1 - \xi) + (A_2 - A_1) \cdot \frac{(1+\xi)}{2L}$$

$$m = \int_{-1}^{1} \frac{1}{2} \begin{bmatrix} (1-\xi) \\ (1+\xi) \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} (1-\xi) & (1+\xi) \end{bmatrix}\ \rho\ A(\xi)\ |J|\ d\xi$$

$$m = \frac{1}{8}\ A_1\rho\ L \int_{-1}^{1} (1-\xi) \cdot \begin{bmatrix} (1-\xi)^2 & (1-\xi^2) \\ (1-\xi^2) & (1+\xi)^2 \end{bmatrix} d\xi + \frac{1}{8}\ (A_2 - A_1)\rho\ L \int_{-1}^{1} (1+\xi) \cdot \begin{bmatrix} (1-\xi)^2 & (1-\xi^2) \\ (1-\xi^2) & (1+\xi)^2 \end{bmatrix} d\xi$$

$$m = \frac{1}{8}\ A_1\rho\ L \cdot \begin{bmatrix} 4 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} + \frac{1}{8}\ (A_2 - A_1)\rho\ L \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 4 \end{bmatrix}$$

$$m = A_1\rho\ L \cdot \begin{bmatrix} 2 & 1/24 \\ 1/24 & 1/24 \end{bmatrix} +\ A_2\rho\ L \begin{bmatrix} 1/24 & 1/24 \\ 1/24 & 2 \end{bmatrix}$$

5 Assignment 10.5

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

According to the *slide 16* the mass matrix of a 3D two-noded bar is:

$$m = \frac{\rho\ A\ L}{2} \cdot I_6 = \frac{\rho\ A\ L}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$