Computational Structural Mechanics and Dynamics

Masters in Numerical Methods

Assignment 10

Solid and Structural Dynamics

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1 Problem 1

In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

To solve for an undamped case we consider $F = 0$

$$ku + m\ddot{u} = 0 \tag{1}$$

The equation of this system is given by $u = u_0 sin(\omega t + \phi)$, u_0 being the amplitude, and ω is the natural frequency of vibration, whose value is given by

$$\omega = \sqrt{\frac{k}{m}} \tag{2}$$

Now for the case of nonzero forces we can see that,

$$ku + m\ddot{u} = F \tag{3}$$

This is an non-homogeneous differential equation, the solution is given by

$$u(t) = u_c(t) + u_p(t) \tag{4}$$

Here, u_c is the complimentary solution of the free undamped part, and u_p is the particular solution.

To solve the particular solution we solve

$$ku_p + m\ddot{u}_p = F \tag{5}$$

with $u_p = c$ a constant

That gives us $u_p = \frac{F}{k}$, therefore we can say that

$$u(t) = \frac{F}{k} + u_0 sin(\omega t + \phi) \tag{6}$$

The above equation shows that F does not affect the system as the particular solution which is dependent of F is independent of the frequency (ω). It is simply an offset value to the solution. The variation in F may change the value of $u(t)$ but it will affect the displacement in the same way irrespective of the frequency.

2 Problem 2

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A.

The maximum displacement will be at $x = \frac{L}{2}$ and it is given by

$$u_{max} = \frac{FL^3}{192EI} \tag{7}$$

If we consider a square beam, of side a we can write the moment of inertia as follows

$$I = \frac{bh^3}{12} = \frac{a^4}{12} \tag{8}$$

substituting in max displacement equation we get

$$u_{max} = \frac{FL^3}{16Ea^4} \tag{9}$$

Therefore the effective stiffness is

$$k = \frac{F}{\frac{FL^3}{16Ea^4}} = \frac{16Ea^4}{L^3} \tag{10}$$

The frequency is given my

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16Ea^4}{mL^3}} = \frac{4a^2}{L}\sqrt{\frac{E}{mL}} \tag{11}$$

3 Problem 3

Use the expression on slide 18 to derive the mass matrix of slide 17.

The expression is

$$\boldsymbol{m} = \int \boldsymbol{N^T N} \rho dV \tag{12}$$

We can write this as

$$m_{ij} = \rho AL \int_0^1 N_i N_j d\eta \tag{13}$$

Now we can introduce shape functions for this problem $N_1 = 1 - \eta$ and $N_2 = \eta$.
The equation becomes

$$\boldsymbol{M} = \rho AL \int_0^1 \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} d\eta \tag{14}$$

Now we can calculate these individually, as following

$$\int_0^1 N_1^2 d\eta = \int_0^1 (1 - 2\eta + \eta^2) d\eta = \frac{1}{3} \tag{15}$$

$$\int_0^1 N_1 N_2 d\eta = \int_0^1 (\eta - \eta^2) d\eta = \frac{1}{6} \tag{16}$$

$$\int_0^1 N_1^2 d\eta = \int_0^1 (\eta^2) d\eta = \frac{1}{3} \tag{17}$$

The final mass matrix turns out to be

$$M = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{18}$$

4 Problem 4

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

The variation in the area can be defined as follows

$$A(\eta) = \sum N_i(\eta) A_i \tag{19}$$

The shape functions for a linear iso-parametric element are given by $N_1 = \frac{1}{2}(1-\eta)$ and $N_2 = \frac{1}{2}(1+\eta)$

Therefore the mass matrix is given by the following expression,

$$\boldsymbol{M} = \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\eta) \\ \frac{1}{2}(1+\eta) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\eta) & \frac{1}{2}(1+\eta) \end{bmatrix} \rho(\frac{A_1}{2}(1-\eta) + \frac{A_2}{2}(1+\eta)|J|d\eta \tag{20}$$

The Jacobian $|J| = \frac{L}{2}$,Further simplifying 20 we get

$$\boldsymbol{M} = \int_{-1}^{1} A_1 \begin{bmatrix} -(\eta-1)^3 & (\eta-1)^2(\eta+1) \\ (\eta-1)^2(\eta+1) & (\eta+1)^2(\eta-1) \end{bmatrix} + A_2 \begin{bmatrix} (\eta-1)^2(\eta+1) & -(\eta+1)^2(\eta-1) \\ -(\eta+1)^2(\eta-1) & (\eta+1)^3 \end{bmatrix} d\eta \tag{21}$$

Finally performing the integration in the given limits we obtain

$$\boldsymbol{M} = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix} \tag{22}$$

5 Problem 5

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

Let the density be ρ, area A and length L. Therefore the total mass is ρAL and since we have two nodes the mass matrix will take the form

$$M = \frac{\rho AL}{2} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{23}$$