Master's Degree Numerical
Methods in Engineering

UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

Computational Structural Mechanics and Dynamics

Homework 1: Direct Stiffness Method

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February 17^{th}, 2020
Academic Year 2019-2020

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Assignment 2.1

On "FEM Modelling: Introduction"

1. Identify the symmetry and antisymmetry lines in the two-dimensional problems illustrated in the figure. They are:
   (a) a circular disk under two diametrically opposite point forces (the famous "Brazilian test" for concrete)
   (b) the same disk under two diametrically opposite force pairs
   (c) a clamped semiannulus under a force pair oriented as shown
   (d) a stretched rectangular plate with a central circular hole.
   (e) and (f) are half-planes under concentrated loads.
2. Having identified those symmetry/antisymmetry lines, state whether it is possible to cut the complete structure to one half or one quarter before laying out a finite element mesh. Then draw a coarse FE mesh indicating, with rollers or fixed supports, which kind of displacement BCs you would specify on the symmetry or antisymmetry lines.

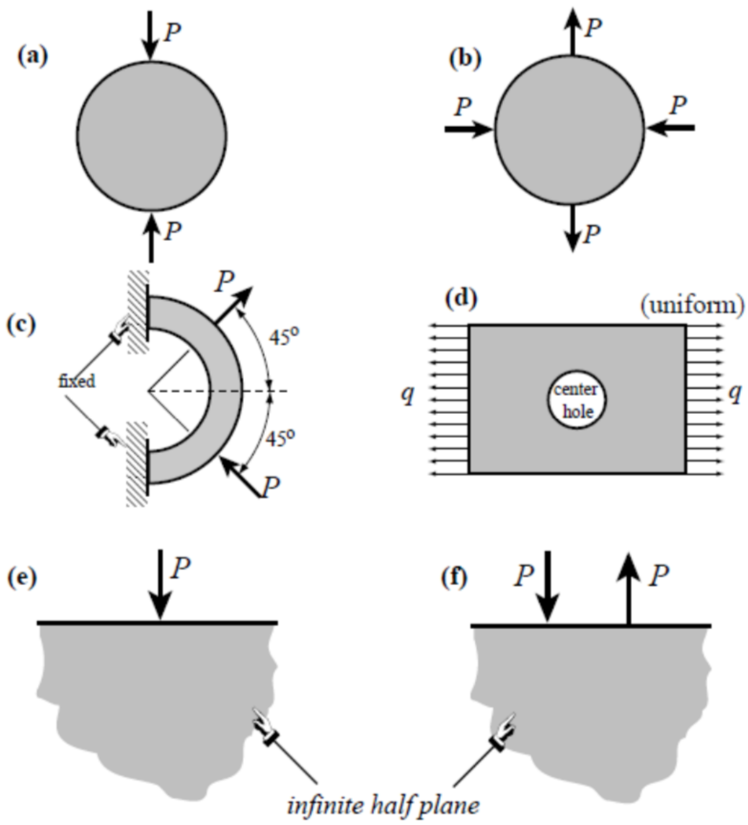


Figure 1: General formulation of displacement based strains.

Assignment 2.2

Explain the difference between *Verification* and *Validation* in the context of the FEM-Modelling procedure.

Assignment 2.3

On *Variational Formulation*:

1. A tapered bar element of length l and areas A_i and A_j with A interpolated as

$$A = A_i \cdot (1 - \xi) + A_j \cdot \xi$$

and constant density ρ rotates on a plane at uniform angular velocity ω (rad/sec) about node i. Taking axis x along the rotating bar with origin at node i, the centrifugal axial force is $q(x) = \rho A \omega^2 \cdot x$ along the length in which x is the longitudinal coordinate $x = x^e$. Find the consistent node forces as functions of ρ, A_i, A_j, ω and l, and specialize the result to the prismatic bar $A = Ai = Aj$.

1 Resolution

1.1 Assignment 2.1: First task

The symmetry and antisymmetry lines are sketched in the Figure 2.

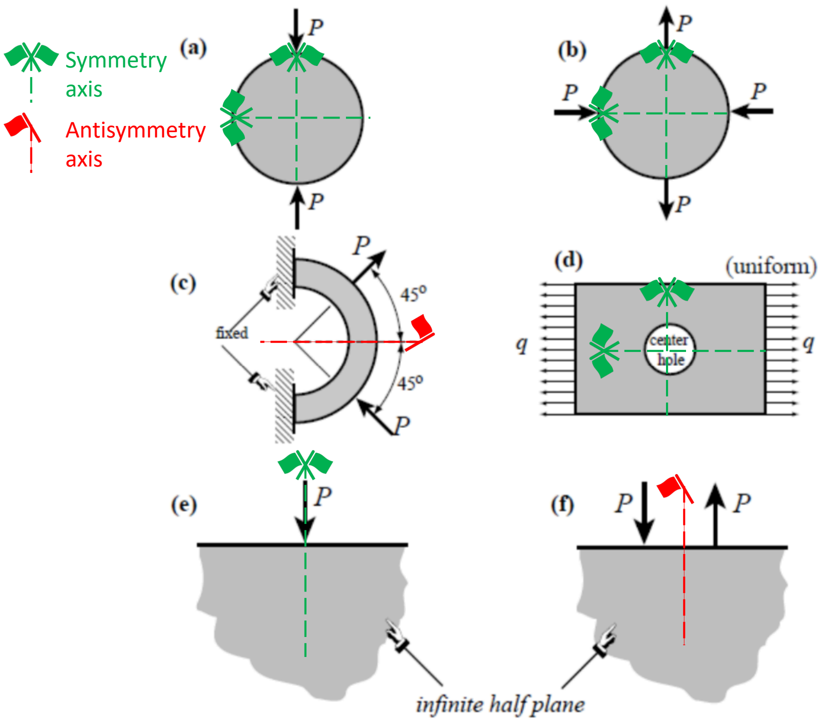


Figure 2: Symmetry and Antisymmetry axis denoted over the structures.

1.2 Assignment 2.1: Second task

The structures in the Figure 2 can be simplified as in half and quarters:

(a) Circular disk 1: **Quarter**

(b) Circular disk 2: **Quarter**

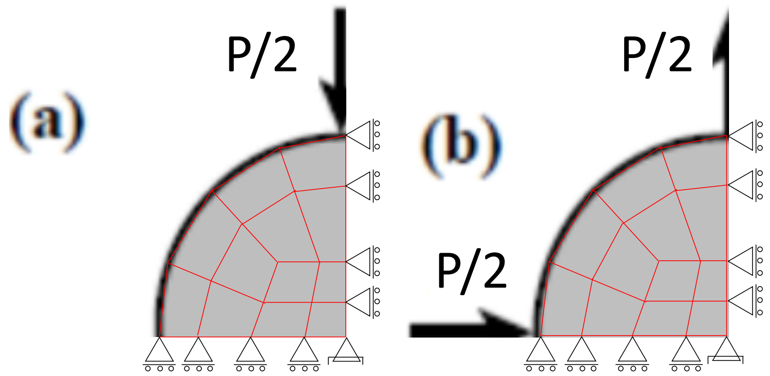
(c) Semiannulus: *Half*

(d) Rectangular plate: *Half*

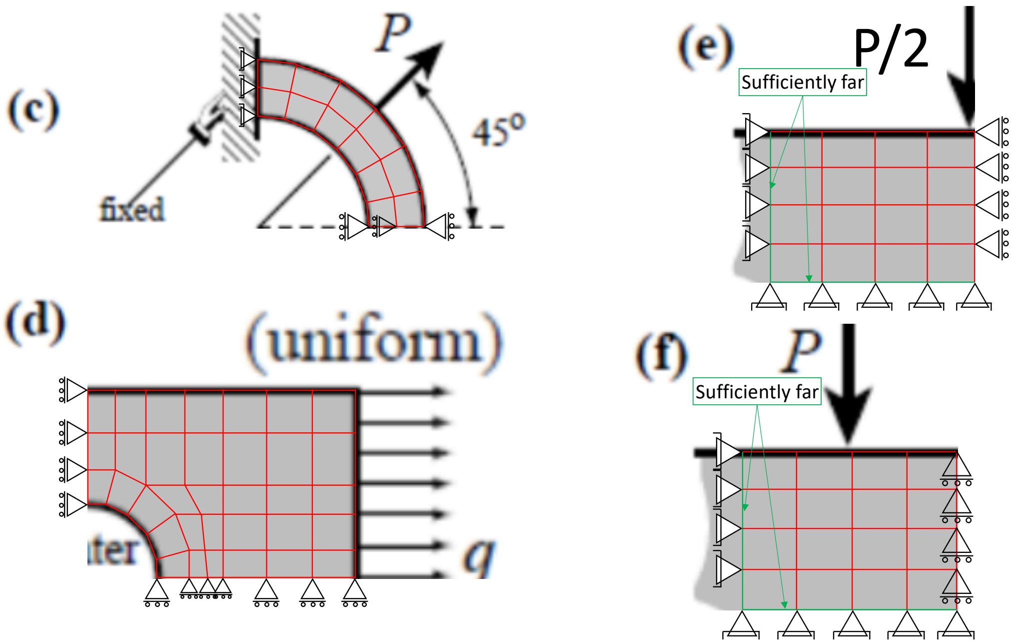
(e) Infinite half plane 1: *Half*

(f) Infinite half plane 2: *Half*

Afterwards, a coarse mesh with the boundary conditions for each of the models is required and scratched in Figure 3. In case (d) the distributed load q is applied discretely in the nodes of the corresponding boundary (right boundary).



(a) Mesh and boundary conditions for cases (a) and (b).



(b) Mesh and boundary conditions for cases (c), (d), (e) and (f).

Figure 3: Coarse mesh and boundary conditions for the different models.

1.3 Assignment 2.2

*Explain the difference between **Verification** and **Validation** in the context of the FEM-Modelling procedure.*

When solving an engineering problem, the reality needs to be modelled whether analytic or numerically, therefore a first approach generally called as *abstraction* is made to generate a first **conceptual model** (for example abstracting a bridge to a beam).
The next step is to start what is known as ***Verification*** which includes the development of the mathematical model governing the physics of the problem (for example the implementation of Euler-Bernoulli bending section or the implementation of Timoshenko theory to the bending of the section) to a computational code which will require to be ***verificated*** in order to prove that what we expected to find a simple case will appear in the model (for example we can probe that a beam uniformly load with vertical fixities on the edges will have a displacement on the center of $\delta = (5q \cdot l^4)/(384 \cdot EI)$).
Once the model correctly represents the physics of a simple problem, the model is verificated and the complex model of interest can be analysed using this computational model. With the results of the model representing the problem of interest, we need now to perform a ***Validation*** comparing the results of the model of interest with a physical representation of the problem of interest. For this part two main strategies can be used, the first is to perform this comparison between the computational model and an existing structure that has an equivalence with the structure we are interested in (for example other similar existing bridges) or the other possibility is to perform a real physical representation in terms of an scaled model in the lab to measure the stresses and strains.

The main characteristics previously mentioned are summarized in Figure 4.

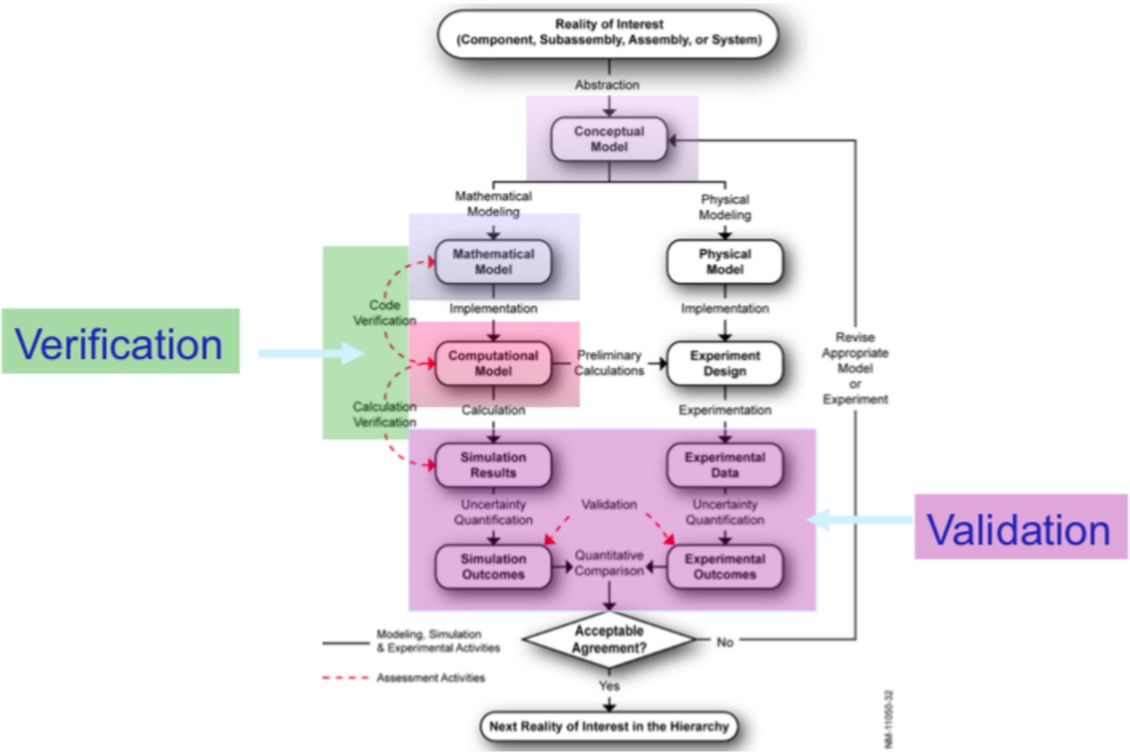


Figure 4: Verification and validation process in FEM.

1.4 Assignment 2.3

A tapered bar element of length l and areas A_i and A_j with A interpolated as $A = A_i \cdot (1 - \xi) + A_j \cdot \xi$ and constant density ρ rotates on a plane at uniform angular velocity ω (rad/sec) about node i. Taking axis x along the rotating bar with origin at node i, the centrifugal axial force is $q(x) = \rho A \omega^2 \cdot x$ along the length in which x is the longitudinal coordinate $x = x^e$.
Find the consistent node forces as functions of ρ, A_i, A_j, ω and l, and specialize the result to the prismatic bar $A = Ai = Aj$. An interpretation of the problem is represented in Figure 5.

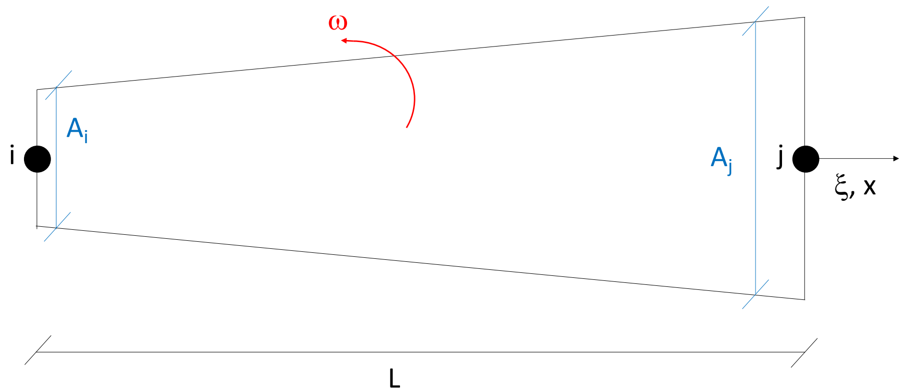


Figure 5: Varying section rotating bar.

Given the fact that the problem does not have a graphical representation, some assumptions must be made. For example, I interpret that the coordinates $x,\ \xi$ are the same, therefore the area of the element can be written as $A(\xi) = A(x)$. Also, the length of the element L is taken as a unit to match the equation $A = A_i \cdot (1 - x) + A_j \cdot x$ with $0 \leq x \leq L,\ L = 1$.

To find the forces applied to each of the nodes in the variational scheme, the external work equation is used to find the load for each of the nodes:

$$f^{ext} = \int_0^L q(x) \cdot \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \cdot L \cdot d\xi$$

Remembering the definition of the ξ coordinate:

$$\xi = \frac{x - x_1}{L}$$

In the case of analysis we can consider $x_1 = 0$, then $x = \xi \cdot L$, then replacing the general equation for external work considering the expression for $q(x)$ and re-writing everything in term of the ξ variable, we obtain the following equations:

$$f^{ext} = \int_0^1 \rho \cdot A(\xi) \cdot \omega^2 \cdot \xi \cdot L \cdot \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \cdot L \cdot d\xi = \rho \cdot \omega^2 \cdot L^2 \int_0^1 (A_i \cdot (1-\xi) + A_j \cdot \xi) \cdot \xi \cdot \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \cdot d\xi$$

$$\rho \cdot \omega^2 \cdot L^2 \cdot A_i \int_0^1 \begin{bmatrix} \xi \cdot (1 - 2\xi + \xi^2) + (A_j/A_i) \cdot (\xi^2 - \xi^3) \\ \xi^2 \cdot (1-\xi) + (A_j/A_i) \cdot \xi^3 \end{bmatrix} \cdot d\xi = \rho \cdot \omega^2 \cdot L^2 \begin{bmatrix} (\frac{1}{2} - \frac{2}{3} + \frac{1}{4}) + \frac{A_j}{A_i} \cdot (\frac{1}{3} - \frac{1}{4}) \\ (\frac{1}{3} - \frac{1}{4}) + \frac{A_j}{A_i} \cdot \frac{1}{4} \end{bmatrix}$$

$$f^{ext} = \frac{\rho \cdot \omega^2 \cdot L^2}{12} \begin{bmatrix} 1 + \frac{A_j}{A_i} \\ 1 + \frac{3 \cdot A_j}{A_i} \end{bmatrix}$$

From the above expression the vector shows the load f_i in node i and f_j in node j respectively. If $A_i = A_j$ the expression of the forces are:

$$f^{ext} = \frac{\rho \cdot \omega^2 \cdot L^2}{6} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2 Conclusions

The three tasks solved in this assignment presented different features of interest when using numerical methods in engineering. In the case of the *Assignment 2.1 task one* and *Assignment 2.1 task two* the importance of analysing whether the structures present or not a symmetry or antisymmetry axis to reduce at least by a half the system of interest will result in at least a 50 % time saving. In the *Assignment 2.2* it was asked to explain the concepts of ***verification*** and ***validation*** that are two important checkpoints we need to perform in order to ensure the quality of results we are obtaining, in the first case to verify the model we are using is being applied correctly, generally it is done for simpler cases, and afterwards, the results of the model of interest are compared with a relevant reference. Finally, in *Assignment 2.3* an example of a spinning bar was used to determine the loads in the nodes in the framework of variational analysis.