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## Assignment 2 - FEM Modelling (introduction) and Variational Formulation

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### Assignment 2.1

1

The graphics below shows the Symmetric Lines (**SL**) and Anti-Symmetric Line (**ASL**) for each proposed case.

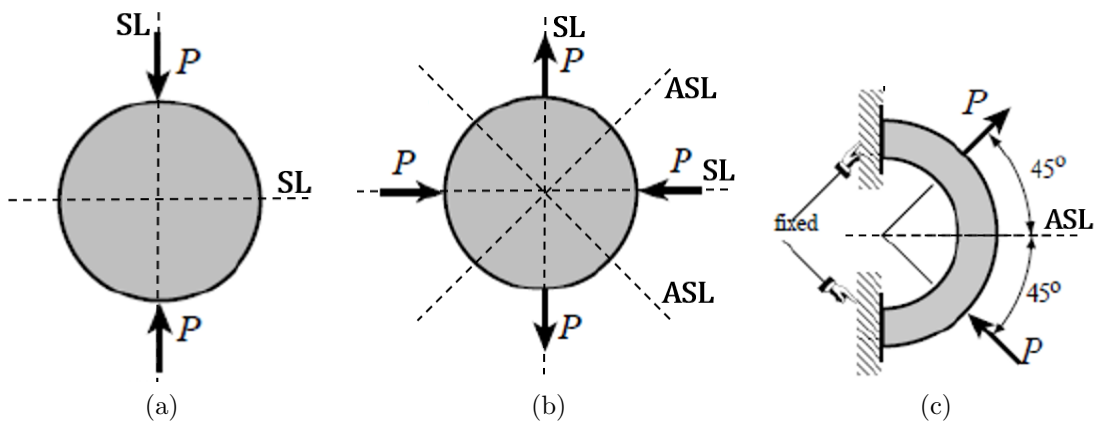


Figure 1: Symmetric and Anti-Symmetric lines for cases (a),(b) and (c).

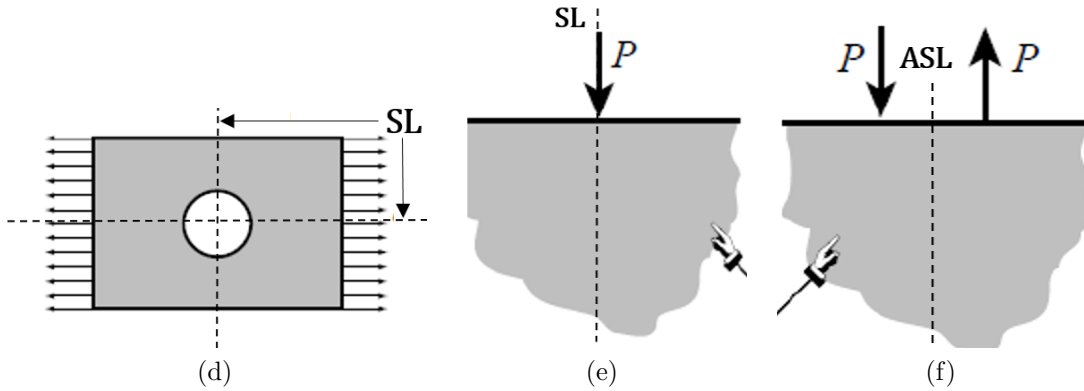


Figure 2: Symmetric and Anti-Symmetric lines for cases (d),(e) and (f).

## 2

The graphics below shows the geometric simplification and appropriated boundary conditions in the mesh nodes in order to explore the symmetry and antisymmetry of each case.

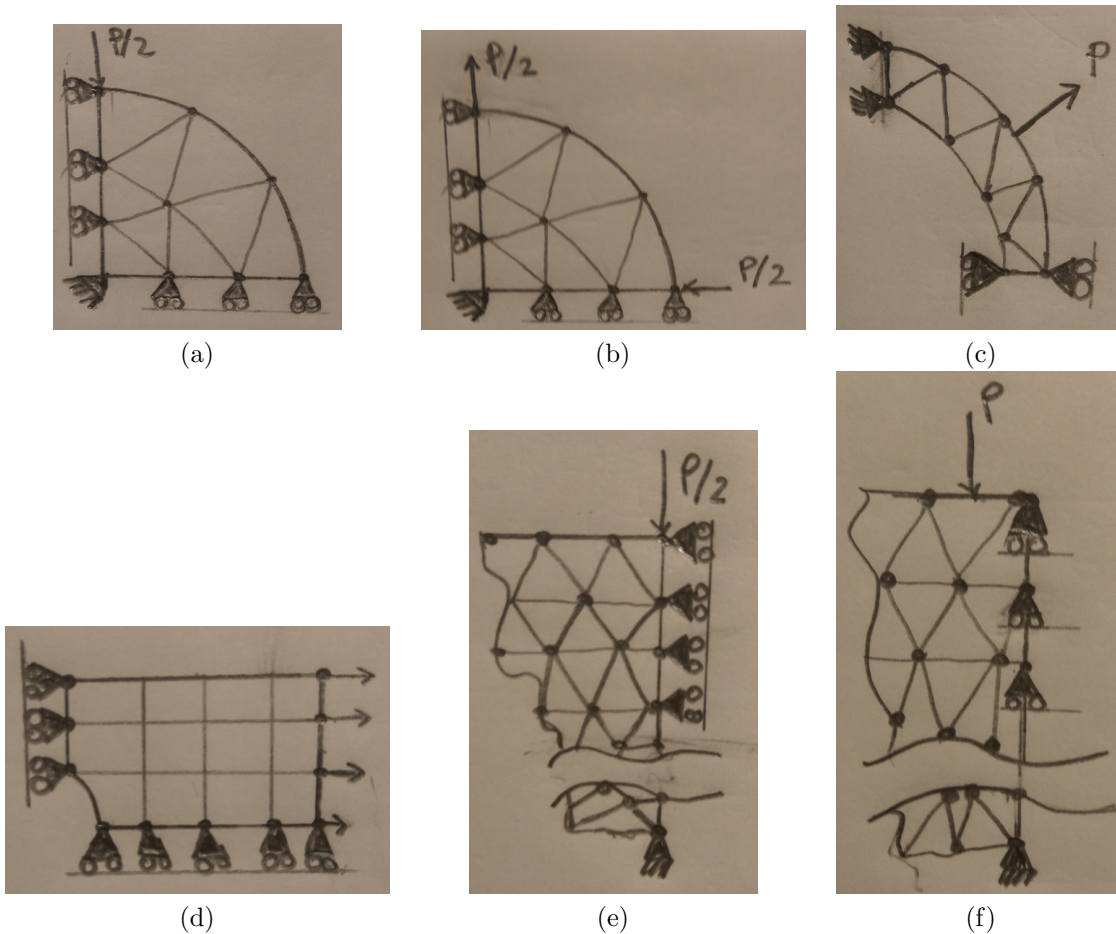


Figure 3: Simplified FEM models.

Notice that for cases (e) and (f), Figure (3) (e) and (f) respectively, a fixed support is imposed in a DOF away from the application force (at "infinity" position). This "infinity" position is a position such that,

for the analysis purposes, the point load effect is not considerable. The case (b) can be further simplified considering its antisymmetric line and implying supports in the DOF's at this line such that the displacement along the line is zero (as made for case (f), in Figure (3) (f)). The result is shown in Figure (4).

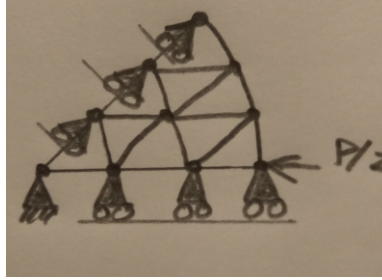


Figure 4: Further simplification of case (b).

## Assignment 2.2

The regions of the geometry, in Figure (5), that should receive some care during meshing, due to high stress gradients, in order to guarantee reasonable results from a Finite Element Analysis are as follows:

- Points D, N and I : Highly Concentrated loads (i.e., point loads).
- Points B, F, M and J: Entrant Corners.

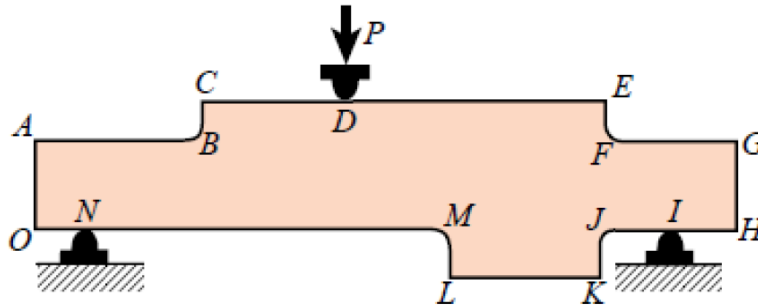


Figure 5: In plane bent plate

## Assignment 2.3

The external work at the element can be written as:

$$W^{(e)} = \int_0^l \mathbf{u}^{(e)} q(x) dx = \int_0^l \mathbf{N} \mathbf{u}^{(e)} q(x) dx = \int_0^l q(x) \mathbf{N}^T \mathbf{u}^{(e)T} dx = \mathbf{u}^{(e)T} \int_0^l q(x) \mathbf{N}^T dx \quad (1)$$

Writing the shape function vector ( $\mathbf{N}^T$ ) and the axial force ( $q(x) = \rho \omega^2 A x$ ) as functions of the natural coordinate  $\xi$  we have:

$$W^{(e)} = \mathbf{u}^{(e)T} \int_0^1 q(\xi) \mathbf{N}(\xi)^T l d\xi = \mathbf{u}^{(e)T} \int_0^1 \rho \omega^2 A(\xi) x(\xi) \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} l d\xi \quad (2)$$

The integral above gives the *consistent nodal forces vector*. By the definition of A ( $(1 - \xi)A_i + \xi A_j$ ) and x in terms of  $\xi$  ( $x = x_1 + l\xi$ , with  $x_1 = 0$  in this case), the vectorial integral for  $f^{(e)}$  above becomes.

$$f^{(e)} = \int_0^1 \rho \omega^2 [(1 - \xi)A_i + \xi A_j] l \xi \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} l d\xi = \rho \omega^2 l^2 \int_0^1 \begin{bmatrix} \xi(1 - \xi)^2 A_i + \xi^2(1 - \xi) A_j \\ \xi^2(1 - \xi) A_i + \xi^3 A_j \end{bmatrix} d\xi \quad (3)$$

The first component of the vector  $f^{(e)}$  is computed as follows:

$$f_1^{(e)} = \rho\omega^2 l^2 \int_0^1 [\xi(1-\xi)^2 A_i + \xi^2(1-\xi) A_j] d\xi$$

$$f_1^{(e)} = \rho\omega^2 l^2 \left[ \left( \frac{\xi^3}{4} - \frac{2\xi^3}{3} + \frac{\xi^2}{2} \right) A_i + \left( \frac{\xi^3}{3} - \frac{\xi^4}{4} \right) A_j \right] \Big|_0^1 = \rho\omega^2 l^2 \left[ \left( \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) A_i + \left( \frac{1}{3} - \frac{1}{4} \right) A_j \right]$$

$$\mathbf{f}_1^{(e)} = \rho \frac{\omega^2 l^2}{12} (\mathbf{A}_i + \mathbf{A}_j) \quad (4)$$

The second component of the vector  $f^{(e)}$  is computed as follows:

$$f_2^{(e)} = \rho\omega^2 l^2 \int_0^1 [\xi^2(1-\xi) A_i + \xi^3 A_j] d\xi$$

$$f_2^{(e)} = \rho\omega^2 l^2 \left[ \left( \frac{\xi^3}{3} - \frac{\xi^4}{4} \right) A_i + \frac{\xi^4}{4} A_j \right] \Big|_0^1 = \rho\omega^2 l^2 \left[ \left( \frac{1}{3} - \frac{1}{4} \right) A_i + \frac{1}{4} A_j \right]$$

$$\mathbf{f}_2^{(e)} = \rho \frac{\omega^2 l^2}{12} (\mathbf{A}_i + 3\mathbf{A}_j) \quad (5)$$

Thus, the *consistent nodal forces vector* becomes:

$$\mathbf{f}^{(e)} = \rho \frac{\omega^2 l^2}{12} \begin{bmatrix} \mathbf{A}_i + \mathbf{A}_j \\ \mathbf{A}_i + 3\mathbf{A}_j \end{bmatrix} \quad (6)$$

When  $A_i = A_j = A$ , the *consistent nodal forces vector* becomes:

$$\mathbf{f}^{(e)} = \rho \frac{\omega^2 l^2}{6} A \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (7)$$

This result provides nodal equilibrium in both nodes. For example, in the case where the first DOF has zero displacement ( $u_1 = 0$ ),  $u_2 = \rho \frac{\omega^2 l^3}{3E}$ , thus the internal force  $F$  (constant for linear elements) is  $F = \rho \frac{\omega^2 l^2}{3} A$ . The reaction for at node one is the integral of the load  $q(x)$  given by  $R = \rho \frac{\omega^2 l^2}{2} A$ . Thus, free body diagram for the 1D- FEM element is as follows.

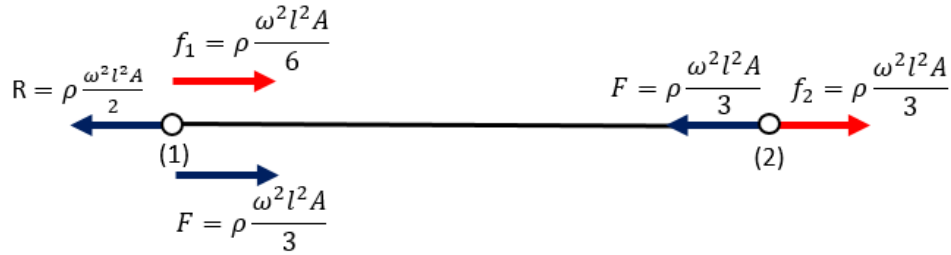


Figure 6: Nodal forces equilibrium.