UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER IN COMPUTATION MECHANICS AND NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 3

by

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1- Introduction

The goal of the assignment is to analyze the Plane Stress Problem and the Linear Triangle element and apply their formulations. A discussion on both subjects was also considered.

2 – Assignment 3.1

2.1 – Part 1

To compute the element stiffness matrix $\boldsymbol{K}^e$ with the data provided in the assignment [1], the following equations were applied [2]:

$$\boldsymbol{K}^{(e)} = \iint_{A^{(e)}} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} t dA \tag{1}$$

where the constitutive matrix $\boldsymbol{D}$ for plane stress analysis is defined as [2]:

$$\boldsymbol{D} = \begin{bmatrix} \frac{E}{1-\upsilon^2} & \frac{\upsilon E}{1-\upsilon^2} & 0 \\ \frac{\upsilon E}{1-\upsilon^2} & \frac{E}{1-\upsilon^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\upsilon)} \end{bmatrix} \tag{2}$$

The matrix $\boldsymbol{B}$ is named as element strain matrix and is defined as [2]:

$$\boldsymbol{B} = \frac{\boldsymbol{1}}{\boldsymbol{2A^{(e)}}} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{3}$$

where the coefficients are related to the coordinates of nodes of the element. They are calculated with the following equation [2]:

$$b_i = y_j - y_i,\ c_i = x_k - x_j; \quad i, j, k = 1,2,3 \tag{4}$$

Considering the assignment data [1] and applying them to Equations 1-4, the element stiffness matrix $\mathbf{K}^e$ can be computed:

$$
\boldsymbol{K}^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}
$$

2.2 – Part 2

The stiffness matrix $\boldsymbol{K}^e$ computed in section 2.1 has rows and columns that if added up, will result in a zero vector. Also, adding up the coefficients of a single row or column will result in a sum equal to zero. As an example, the sum of the 3rd row coefficients, the sum of the 1st, 3rd and 5th rows and the sum of the 2nd, 4th and 6th columns are presented:

<u>Sum of the coefficients from 3rd row:</u>

$$-12.5 + 6.25 +75 -37.5 -62.5 + 31.25 =0$$

<u>Sum of the 1st, 3rd and 5th row vectors:</u>

$$[18.75 \;\; 9.375 \;\; -12.5 \;\; -6.25 \;\; -6.25 \;\; -3.125]$$

$$+ \quad [-12.5 \;\; 6.25 \;\; 75 \;\; -37.5 \;\; -62.5 \;\; 31.25]$$

$$+ \quad [-6.25 \;\; -15.625 \;\; -62.5 \;\; -43.75 \;\; -68.75 \;\; -28.125]$$

$$= [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

<u>Sum of the 2nd, 4th and 4th column vectors:</u>

$$[9.375 \;\; 18.75 \;\; 6.25 \;\; 12.5 \;\; -15.625 \;\; -31.25]^T$$

$$+ \quad [-6.25 \;\; 12.5 \;\; -37.5 \;\; 75 \;\; 43.75 \;\; -87.5]^T$$

$$+ \quad [-3.125 \;\; -31.25 \;\; 31.25 \;\; -87.5 \;\; -28.125 \;\; 118.75]^T$$

$$= [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T$$

The zero values for the sum among either rows or columns or their coefficients must be obtained in order to guarantee the equilibrium between internal and external forces. If the global stiffness matrix does not have zero values for such sums, the values of the displacement and consequently the reaction forces will be miscalculated and resultant force acting on the structure will be different than zero. Therefore, the equilibrium conditions would not be met.

3 – Assignment 3.2

3.1 – Part A

Considering the plane linear Turner Triangle (Figure 1) with thickness t = 1, edge length of a = 1 and material parameters E ≠ 0 and $\upsilon = 0$, its stiffness matrix $\boldsymbol{K}_{triangle}$ can be computed [1]. The Equations 1-4 are applied to obtain the following stiffness matrix for the Turner Triangle $\boldsymbol{K}_{triangle}$:

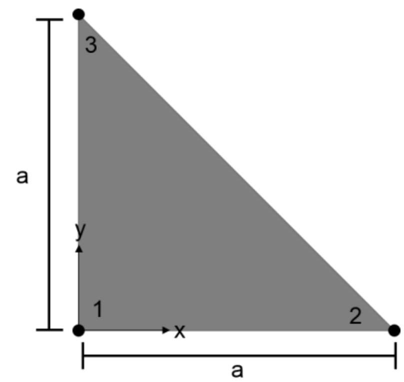

Figure 1. Turner Triangle

$$\boldsymbol{K}_{triangle} = \begin{bmatrix} \frac{3E}{4} & \frac{E}{4} & \frac{-E}{2} & \frac{-E}{4} & \frac{-E}{4} & 0 \\ \frac{E}{4} & \frac{3E}{4} & 0 & \frac{-E}{4} & \frac{-E}{4} & \frac{-E}{2} \\ \frac{-E}{2} & 0 & \frac{E}{2} & 0 & 0 & 0 \\ \frac{-E}{4} & \frac{-E}{4} & 0 & \frac{E}{4} & \frac{E}{4} & 0 \\ \frac{-E}{4} & \frac{-E}{4} & 0 & \frac{E}{4} & \frac{E}{4} & 0 \\ 0 & \frac{-E}{2} & 0 & 0 & 0 & \frac{E}{2} \end{bmatrix}$$

To compute the global stiffness matrix of the truss $\boldsymbol{K}_{bar}$ depicted in Figure 2, the element stiffness matrices $\boldsymbol{K}^{(e)}$ must be computed. The following equations are applied to compute such matrices [3]:

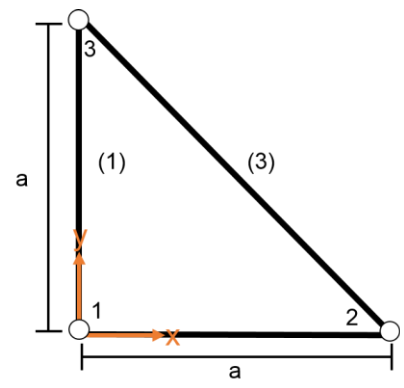

Figure 2. Truss

$$\boldsymbol{K}^{(e)} = (\boldsymbol{T}^{(e)})^T \boldsymbol{K}^{(e)}_{local} \boldsymbol{T}^{(e)} \tag{5}$$

where:

$$\boldsymbol{K}^{(e)}_{local} = \left(\frac{EA}{L}\right)^{(e)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\boldsymbol{T}^{(e)} = \begin{bmatrix} c & -s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & -s & c \end{bmatrix} \tag{7}$$

Considered the data provided in the assignment [1], the element stiffness matrices are:

$$\boldsymbol{K}^{(1)} = (EA_1) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(2)} = (EA_2)\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = \left(\frac{EA_3}{\sqrt{2}}\right)\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Considering $A_1 = A_2 = A$ and the computed stiffness matrices $\boldsymbol{K}^{(1)}$, $\boldsymbol{K}^{(2)}$, $\boldsymbol{K}^{(3)}$, it is possible to assemble the global stiffness matrix for the truss $\boldsymbol{K}_{bar}$:

$$\boldsymbol{K}_{bar} = \begin{bmatrix} EA & 0 & -EA & 0 & 0 & 0 \\ 0 & EA & 0 & 0 & 0 & -EA \\ -EA & 0 & EA + \frac{EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} \\ 0 & 0 & \frac{-EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} \\ 0 & 0 & \frac{-EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} \\ 0 & -EA & \frac{EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} & \frac{-EA_3}{2\sqrt{2}} & EA + \frac{EA_3}{2\sqrt{2}} \end{bmatrix}$$

3.2 – Part B

There are no possible values for $A_1 = A_2 = A$ and $A_3 = A'$ to obtain $\boldsymbol{K}_{bar} = \boldsymbol{K}_{triangle}$, because of the zero coefficients in different positions in both matrices. Although, to make them as similar as possible A would need to assume the value of 3/4 and A' would need to assume the value of $\sqrt{2}/2$.

3.3 – Part C

The stiffness matrices $\boldsymbol{K}_{bar}$ and $\boldsymbol{K}_{triangle}$ are not equivalent, because the truss bar is a 1D structure and the Turner Triangle is a 2D structure. The truss bar only resists displacement along its length and the Turner Triangle resists to displacement throughout its whole midplane. Therefore, their stiffness matrices cannot be the same and would provide different displacement fields under the same load case.

3.4 – Part D

Applying the same procedure as in section 3.1 to compute $\boldsymbol{K}_{triangle}$, but considering $\upsilon \neq 0$, the new $\boldsymbol{K}_{triangle}$ takes the following form:

$$\boldsymbol{K}_{triangle} =$$

$$= \begin{bmatrix} \frac{E}{1-\upsilon^2}+\frac{E}{2(1+\upsilon)} & \frac{\upsilon E}{1-\upsilon^2}+\frac{E}{2(1+\upsilon)} & -\frac{E}{1-\upsilon^2} & -\frac{E}{2(1+\upsilon)} & -\frac{E}{2(1+\upsilon)} & -\frac{\upsilon E}{1-\upsilon^2} \\ \frac{\upsilon E}{1-\upsilon^2}+\frac{E}{2(1+\upsilon)} & \frac{E}{1-\upsilon^2}+\frac{E}{2(1+\upsilon)} & -\frac{\upsilon E}{1-\upsilon^2} & -\frac{E}{2(1+\upsilon)} & -\frac{E}{2(1+\upsilon)} & -\frac{E}{1-\upsilon^2} \\ -\frac{E}{1-\upsilon^2} & -\frac{\upsilon E}{1-\upsilon^2} & \frac{E}{1-\upsilon^2} & 0 & 0 & \frac{\upsilon E}{1-\upsilon^2} \\ -\frac{E}{2(1+\upsilon)} & -\frac{E}{2(1+\upsilon)} & 0 & \frac{E}{2(1+\upsilon)} & \frac{E}{2(1+\upsilon)} & 0 \\ -\frac{E}{2(1+\upsilon)} & -\frac{E}{2(1+\upsilon)} & 0 & \frac{E}{2(1+\upsilon)} & \frac{E}{2(1+\upsilon)} & 0 \\ -\frac{\upsilon E}{1-\upsilon^2} & -\frac{E}{1-\upsilon^2} & \frac{\upsilon E}{1-\upsilon^2} & 0 & 0 & \frac{E}{1-\upsilon^2} \end{bmatrix}$$

Comparing the first $\boldsymbol{K}_{triangle}$ (considering $\upsilon=0$) with the second $\boldsymbol{K}_{triangle}$ (considering $\upsilon \neq 0$), it is possible to point out that the arguments in the matrix considering $\upsilon \neq 0$ are greater (absolute values) than the arguments in the matrix considering $\upsilon=0$ for the same value of Young Modulus E. Under the same load case, the nodal displacements for the stiffness matrix considering $\upsilon \neq 0$ will be smaller, showing that $\upsilon \neq 0$ offers more resistance to displacements. Such observation is coherent, since when $\upsilon=0$ the strains ε_x and ε_y are also considered zero. Therefore, the internal energy that could be stored by the body is reduced and the stiffness matrix, which is derived from the internal energy, has also reduced arguments [3]. In such manner, under the same load case, the nodal displacements would be larger considering $\upsilon=0$.

4 – Discussion on Plane Stress Problem

The Plane Stress Problem is a simplification from a 3D problem to a 2D problem applied to prismatic structures. Specifically to structures which have 2 dominant dimensions (sufficiently large compared to the third dimension) and the midplane formed by the 2 dominant dimensions represent the behavior of structure along the third dimension (thickness). To assure such assumption, the loads must be applied on the midplane formed by the 2 dominant dimensions, not perpendicular to it [2]. Such simplification of the 3D problem provides a reduction of the computational time when FEM is employed to solve the problem. Another important advantage is the facilitated discretization of the 2D domain compared to the initial 3D domain. With a 2D domain, more options on how to build the mesh are available, such as an unstructured mesh with quadrilateral elements. Also, the mesh quality of such 2D domain can be improved when compared to the 3D mesh of the initial domain, because the smallest dimension (thickness) would not be considered. The mesh quality can be degenerated when a certain dimension of the element is much smaller than the others.

5 – Discussion on Linear Triangle Element

The Linear Triangle element, also known as Turner Triangle, has a simple formulation which allows its incorporation in finite element formulation with ease. Such feature made the Linear Triangle very useful, especially because its geometry enables more efficient discretization of complex domain geometries when compared to quadrilateral elements. Also, the linear triangle element is more suitable for adaptive mesh refinement due to its geometrical features [2]. Nevertheless, the linear triangle has the drawback of providing constant strain and stress fields, requiring mesh refinement where more accurate results are needed.

6 – References

[1] – Assignment-3, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[2] – Oñate, E., Díez, P., Zárate, F., Larese, A., “Introduction to the Finite Element Method”, 2008.

[3] – Presentation “The Plane Stress Problem”, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.