CSMD

Assignment - 03

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3.1

Given, for an isotropic elastic material, Lame constants λ & μ are given by,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu = G = \frac{E}{2(1+\nu)}$$

1/ We have

$$\mu = \frac{E}{2(1+\nu)}$$

$$\Rightarrow E = 2\mu(1+\nu) \quad \text{--- (1)}$$

And,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

From (1), we have,

$$\lambda = \frac{2\mu\cancel{(1+\nu)}\nu}{\cancel{(1+\nu)}(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)}$$

$$\Rightarrow \lambda(1-2\nu) = 2\mu\nu$$

$$\Rightarrow \lambda - 2\nu\lambda = 2\mu\nu$$

$$\Rightarrow 2\nu(\lambda+\mu) = \lambda$$

$$\Rightarrow \boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}} \quad \text{--- (2)}$$

Substituting the value of ② in ①, we get

$$E = 2\mu\left(1 + \frac{\lambda}{2(\lambda+\mu)}\right)$$

$$\Rightarrow E = \cancel{2}\mu\left(\frac{2\lambda + 2\mu + \lambda}{\cancel{2}(\lambda+\mu)}\right)$$

$$\Rightarrow \boxed{E = \mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)} \quad -③$$

2/ The elastic matrix for a plane stress problem is given by

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix}$$

Substituting the values of E & ν obtained from eqn ② & ③ into the above equation, we get the following terms—

$$\frac{E}{1-\nu^2} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \times \frac{1}{1-\left[\frac{\lambda}{2(\lambda+\mu)}\right]^2}$$

$$= \frac{\mu(3\lambda+2\mu)}{\cancel{(\lambda+\mu)}} \times \frac{4(\lambda+\mu)^{\cancel{2}}}{4(\lambda+\mu)^2 - \lambda^2}$$

$$= \frac{\mu(3\lambda+2\mu) \times 4(\lambda+\mu)}{(2(\lambda+\mu)+\lambda)(2(\lambda+\mu)-\lambda)}$$

$$= \frac{\mu\cancel{(3\lambda+2\mu)} \times 4(\lambda+\mu)}{\cancel{(3\lambda+2\mu)}(\lambda+2\mu)}$$

$$= \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}$$

$$\frac{1-\nu}{2} = \frac{1}{2} \times \left[1 - \frac{\lambda}{2(\lambda+\mu)}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda+2\mu}{2(\lambda+\mu)}\right] = \frac{\lambda+2\mu}{4(\lambda+\mu)}$$

Substituting these values in the plane stress elastic matrix, we get-

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)} \end{bmatrix}$$

* The elastic matrix for a plane-strain problem can be expressed as follows—

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

Here,

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{2\mu\cancel{(1+\nu)}(1-\nu)}{\cancel{(1+\nu)}(1-2\nu)} \quad [\text{From eq}^n \text{ 1}]$$

$$= \frac{2\mu(1-\nu)}{(1-2\nu)}$$

$$= 2\mu \times \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{1-\frac{2\lambda}{2(\lambda+\mu)}}$$

$$= \cancel{2\mu} \times \frac{2\lambda+2\mu-\lambda}{\cancel{2\lambda}+\cancel{2\mu}-\cancel{2\lambda}}$$

$$= \lambda+2\mu$$

$$\frac{\nu}{1-\nu} = \frac{\frac{\lambda}{2(\lambda+\mu)}}{1-\frac{\lambda}{2(\lambda+\mu)}} *$$

$$= \frac{\lambda}{\cancel{2(\lambda+\mu)}} \times \frac{\cancel{2(\lambda+\mu)}}{\lambda+2\mu}$$

$$= \frac{\lambda}{\lambda+2\mu}$$

& $\frac{1-2\nu}{2(1-\nu)} = \frac{2-2\nu-1}{2(1-\nu)}$

$$= \frac{2(1-\nu)-1}{2(1-\nu)}$$

$$= 1-\frac{1}{2(1-\nu)}$$

$$= 1-\frac{1}{2\left(1-\frac{\lambda}{2(\lambda+\mu)}\right)}$$

$$= 1-\frac{2(\lambda+\mu)}{2(\lambda+2\mu)}$$

$$= \frac{2\lambda+4\mu-2\lambda-2\mu}{2(\lambda+2\mu)}$$

$$= \frac{\mu}{\lambda+2\mu}$$

Substituting these values in plain strain elastic matrix we get,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \lambda+2\mu\begin{bmatrix}1 & \frac{\lambda}{\lambda+2\mu} & 0\\ \frac{\lambda}{\lambda+2\mu} & 1 & 0\\ 0 & 0 & \frac{\mu}{\lambda+2\mu}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix}$$

3/ From the above matrix equation of plain strain, we have,

$$E_{strain} = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

$$= \begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

$$= \lambda\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} + \mu\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \boxed{E_{strain} = E_{\lambda} + E_{\mu}}$$

4/1

$$E_{\lambda} = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

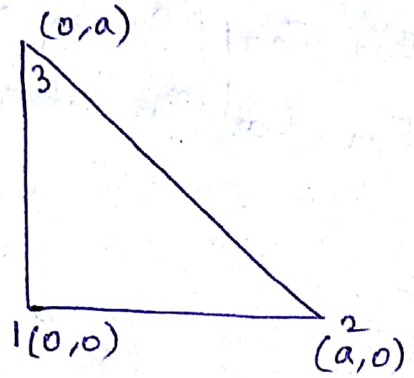
$$= \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ [given value of λ]

&, $$E_{\mu} = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [given value of μ]

3.2

Given, a plane triangular domain of thickness h. with horizontal & vertical length, $a = 1$ & thickness $h = 1$.



∴ Here, $x_1 = 0$ $\quad x_2 = a$ $\quad x_3 = 0$

$y_1 = 0$ $\quad y_2 = 0$ $\quad y_3 = a$

∴ Area, $2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$

$= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$\Rightarrow A = a/2 = 1/2$

For plain-stress model,

$$E_{stress} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Given, $\nu = 0$, initially

$$\therefore E_{stress} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Now,

Stiffness matrix,

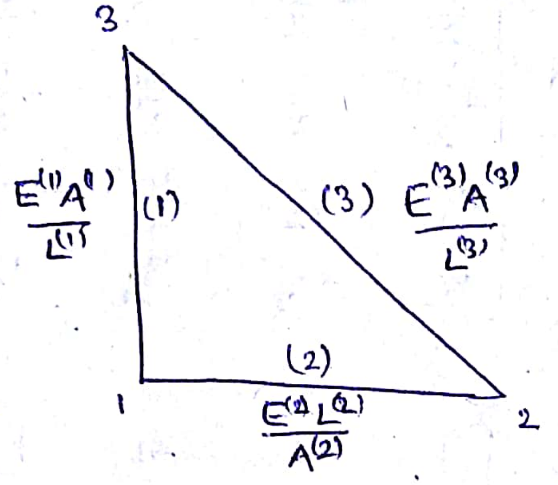
$$K^e_{tri} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{E}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{6\times 3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}_{3\times 3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}_{3\times 6}.$$

On solving, we get

$$K^e_{tri} = \frac{E}{2}\begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, for a set of 3 bar elements,



Assuming, $\cancel{E_1A_1 = E_2A_2 = E_3A_3}$

$\& \; L_1 = L_2 = a = 1$

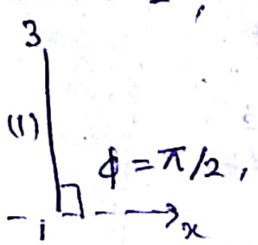
$\therefore L_3 = \sqrt{2}\,a = \sqrt{2}$

For a bar element, stiffness matrix,

$$K^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & c^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$s = \sin\phi, \; c = \cos\phi$

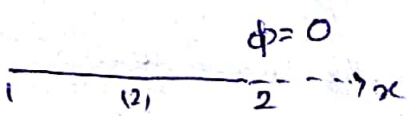
For element '1',



$$\therefore K^{(1)} = \frac{E^{(1)}A^{(1)}}{1} \begin{array}{c} \begin{array}{cccc} 1 & 1 & 3 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 1 \\ 3 \\ 3 \end{array}$$

$s = 1, \; c = 0$

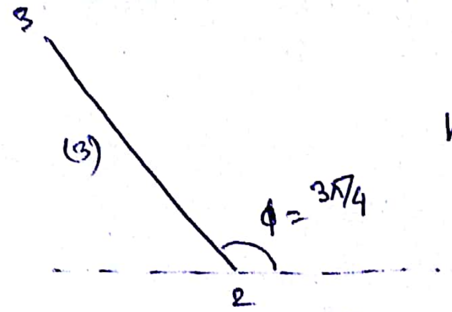
For element '2',



$$K^{(2)} = \frac{E^{(2)}A^{(2)}}{1} \begin{array}{c} \begin{array}{cccc} 1 & 1 & 2 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \end{array}$$

$s = 0, \; c = 1$

For element '3',



$$K^{(3)} = \frac{E^{(3)}A^{(3)}}{\cancel{2}\sqrt{2}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$s = \frac{\sqrt{2}}{2}$, $c = -\frac{\sqrt{2}}{2}$

$$\Rightarrow K^{(3)} = \frac{EA_3}{2\sqrt{2}} \begin{array}{c} \begin{array}{cccc} 2 & 2 & 3 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 2 \\ 3 \\ 3 \end{array}$$

$\therefore$ Global stiffness matrix,

$$K_{bar} = K_1 + K_2 + K_3$$

$$\Rightarrow K_{bar} = E \begin{bmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_2 & 0 & \left(A_2 + \frac{A_3}{2\sqrt{2}}\right) & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix}$$

2/ It is clear from the stiffness matrices of the triangular & bar model, that K_{bar} & K_{tri} are completely different & share no similarity.

We can make them more equivalent by taking $A_1 = A_2 = \frac{1}{2}$ & $A_3 = \sqrt{2}$, but not exact.

3/ The stiffness matrix K_{tri} is able to compute the stresses & strains within the boundary of the closed triangular domain in the problem.

But, when bar elements are considered, only displacements along the axial directions are considered and the points within the boundary of the triangular domain are unaccounted for.

4/ If $\nu \neq 0$, then shear strains will come into the picture which was not the case when $\nu = 0$.(only diagonal ~~elements~~ components were present). This will make K_{tri} much more different than K_{bar}. Overall, we find that K_{tri} is more efficient in analyzing such models compared to bar elements which are effective at only truss problems.