Plane Stress **Assignment 3.1** Plane Strain

$$\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$$

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

Plane strain can be written as:

$$\frac{E^*}{1-\nu^{*2}}\begin{bmatrix}1 & \nu^* & 0\\ \nu^* & 1 & 0\\ 0 & 0 & \frac{1-\nu^*}{2}\end{bmatrix}$$

$$\frac{E^*}{1-\nu^{*2}} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)} \quad 1$$

$$\frac{E^*\,(1-\nu^*)}{2\,(1-\nu^{*2})} = \frac{E\,(1-2\nu)}{(1+\nu)(1-2\nu)\,2} \quad 2$$

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from 1 find E^*

$$\frac{E^*}{(1-\nu^*)(1+\nu^*)} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$E^* = \frac{E\,(1-\nu)(1-\nu^*)(1+\nu^*)}{(1+\nu)(1-2\nu)} \quad \text{— } 3$$

Substitute 3 in 2

$$\frac{\cancel{E}(1-\nu)\cancel{(1-\nu^*)}\cancel{(1+\nu^*)}(1-\nu^*)}{\cancel{2}\cancel{(1-\nu^{*2})}\cancel{(1+\nu)}(1-2\nu)} = \frac{\cancel{E}}{\cancel{(1+\nu)}\cancel{2}}$$

$$\frac{(1-\nu)(1-\nu^*)}{(1-2\nu)} = 1$$

$$(1-\nu)(1-\nu^*) = 1-2\nu$$

$$1-\nu^* = \frac{1-2\nu}{1-\nu}$$

$$\nu^* = -\frac{1-2\nu}{1-\nu} + 1$$

$$= \frac{2\nu-1}{1-\nu} + \frac{1-\nu}{1-\nu}$$

$$= \frac{-1+2\nu+1-\nu}{1-\nu}$$

$$= \frac{(2\nu-\nu)+(1-1)}{1-\nu}$$

$$\boxed{\nu^* = \frac{\nu}{1-\nu}} \text{ — } 4$$

Substitute 4 in 1

$$\frac{E^*}{1-\nu^{*2}} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*}{1-\frac{\nu^2}{(1-\nu)^2}} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*\,(1-\nu)^2}{(1-\nu)^2-\nu^2} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*\,(1-\nu)^{\cancel{2}}}{\cancel{1-2\nu}+\cancel{\nu^2}-\cancel{\nu^2}} = \frac{E\,\cancel{(1-\nu)}}{(1+\nu)\cancel{(1-2\nu)}}$$

$$\boxed{E^* = \frac{E}{(1-\nu)(1+\nu)} = \frac{E}{1-\nu^2}}$$

Scanned with CamScanner

Assignment 3.1

Plane Stress can be written as:

$$\frac{E^*}{(1+\nu^*)(1-2\nu^*)}\begin{bmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & \frac{1-2\nu^*}{2} \end{bmatrix}$$

$$\frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2} \quad \ldots\ldots (1)$$

$$\frac{E^*(1-2\nu^*)}{2(1+\nu^*)(1-2\nu^*)} = \frac{E(1-\nu)}{(1-\nu^2)2} \quad \ldots\ldots (2)$$

from (1) find E^*

$$E^* = \frac{E(1+\nu^*)(1-2\nu^*)}{(1-\nu^2)(1-\nu^*)} \quad \ldots\ldots (3)$$

Substitute (3) in (2)

$$\frac{\cancel{E}(1\cancel{+\nu^*})(1-2\nu^*)}{\cancel{2}(1\cancel{-\nu^2})(1-\nu^*)(1\cancel{+\nu^*})} = \frac{\cancel{E}(1-\nu)}{(1-\nu^2)\cancel{2}}$$

$$\frac{1-2\nu^*}{1-\nu^*} = 1-\nu$$

$$\frac{\nu^*}{1-\nu^*} = \nu$$

$$\frac{1}{\frac{1}{\nu^*}-1} = \nu$$

$$\frac{1}{\nu^*} = \frac{1}{\nu}+1 = \frac{1+\nu}{\nu}$$

$$\boxed{\nu^* = \frac{\nu}{1+\nu}} \text{ --- } (4)$$

Substitute (4) in (1)

$$\frac{E^*(1+\nu-\nu)\cancel{\frac{1}{1+\nu}}}{(1+\nu+\nu)(1+\nu-2\nu)\cancel{\frac{1}{1+\nu}}\frac{1}{1+\nu}} = \frac{E}{1-\nu^2}$$

$$\frac{E^*(1+\nu)}{(1+2\nu)(1-\nu)} = \frac{E}{1-\nu^2}$$

$$E^* = \frac{E(1-\nu)(1+2\nu)}{(1+\nu)(1-\nu^2)}$$

$$\boxed{E^* = \frac{E(1+2\nu)}{(1+\nu)^2}}$$

Assignment 3.2

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \cdots (1) \qquad \mu = G = \frac{E}{2(1+\nu)} \cdots (2)$$

Obtain E :

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} \cdots\cdots (3)$$

Substitute (4) in (3)

$$E = \frac{\lambda\left(1+\frac{E}{2\mu}-1\right)\left(1-2\left(\frac{E}{2\mu}-1\right)\right)}{\frac{E}{2\mu}-1}$$

$$E = \frac{\lambda\left(\frac{E}{2\mu}\right)\left(\frac{-2E+6\mu}{\cancel{2\mu}}\right)}{\frac{E-2\mu}{\cancel{2\mu}}}$$

$$E = \frac{\lambda E(-2E+6\mu)}{2\mu(E-2\mu)}$$

$$E = \frac{\lambda E(-2E+3\mu)2}{2\mu(E-2\mu)} =$$

$$\mu(E-2\mu) = \lambda(-E+3\mu)$$

$$-\lambda E-\mu E = -2\mu^2-3\lambda\mu$$

$$\boxed{E = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}} \text{ — } (5)$$

Obtain ν

$$\nu = \frac{E}{2\mu}-1 \cdots\cdots (4)$$

Substitute (5) in (4)

$$\nu = \frac{(3\lambda+2\mu)\cancel{\mu}}{(\lambda+\mu)2\cancel{\mu}} - 1$$

$$\nu = \frac{3\lambda+2\mu-2(\lambda+\mu)}{2(\lambda+\mu)}$$

$$\nu = \frac{3\lambda+2\mu-2\lambda-2\mu}{2(\lambda+\mu)}$$

$$\boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}}$$

Assingment 3-3

Linear triangular Element

$$K^e = \int_{\Omega^e} h B^T E B \, d\Omega$$

Bar Element Stiffness matrix

$$K^e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{(e)} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\nu = 0 \qquad \mathbf{E} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad h = a = 1 \qquad \therefore A = \frac{1}{2}$$

$$K^e = \frac{1}{4\,\frac{1}{2}}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boxed{K^e = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}}$$

Stiffness Matrix for Turner triangle

For the model using bar elements

$$\bar{K} = \frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the respective angles, one obtains the elemental stiffnes as $(T^e)^T \bar{K}^e T^e$ that is:

$$K^e = \frac{EA}{l}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Assignment 3.3

$$K_1 = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad K_2 = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = \frac{EA_3}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$A = A_1 = A_2$

$$K^e = \begin{bmatrix} EA & 0 & -EA & 0 & 0 & 0 \\ 0 & EA & 0 & 0 & 0 & -EA \\ -EA & 0 & EA+\frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & -\frac{EA_3}{2\sqrt{2}} & \frac{EA'}{2\sqrt{2}} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} \\ 0 & 0 & -\frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & \frac{EA_3}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} \\ 0 & -EA & \frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} & -\frac{EA'}{2\sqrt{2}} & EA+\frac{EA'}{2\sqrt{2}} \end{bmatrix}$$

Stiffness Matrix for bar model

b)

c) The matrices are not equal because they come from different models. While the turner triangle considers shear stresses, the bar model can only withstand axial stresses.

d) When $\nu \neq 0$, plane stress and plane strain are different ~~to~~ one another.

$$K_{p.stress} = \frac{1}{2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K_{p\,strain} = \frac{1}{2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

An important difference is that one can recover the stess (in a plain strain) or the ~~stress~~ strain (in a plane stess) as a postprocess, while the bars do not consider effects occurring in the third direction.