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MSc. Computational Mechanics Erasmus Mundus

Assignment 3: Plane stress &
Linear triangle problem

Computational Structural Mechanics & Dynamics

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Assignment 3.1

On "The Plane Stress Problem":

In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

1. Find the inverse relations for E, ν in terms of λ, μ.

Solution: We know,

$$\mu = \frac{E}{2(1+\nu)} \quad \text{or} \quad E = 2\mu(1+\nu)$$

Using this in the equation for λ, we get,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu(1+\nu)}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{1-2\nu}$$

Therefore,

$$\lambda(1-2\nu) = 2\mu\nu \implies \lambda - 2\lambda\nu = 2\mu\nu \implies \lambda = 2\nu(\lambda+\mu)$$

Hence, we get,

$$\boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}}$$

Now using this in the equation for E, we get,

$$E = 2\mu\left(1 + \left[\frac{\lambda}{2(\lambda+\mu)}\right]\right)$$

$$E = \mu\left(\frac{2\mu + 2\lambda + \lambda}{\lambda+\mu}\right)$$

Hence, we get,

$$\boxed{E = \mu\left(\frac{2\mu + 3\lambda}{\lambda+\mu}\right)}$$

2. Express the elastic matrix for plane stress and plane strain cases in terms of λ, μ.

Solution: The elastic matrix for plane stress condition is given as,

$$\boldsymbol{E}_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}$$

where $\boldsymbol{E}_\sigma$ denotes the elastic matrix for plane stress condition.

Given that,

$$E = 2\mu(1+\nu)$$

we have,

$$\boldsymbol{E}_\sigma = \frac{2\mu(1+\nu)}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}$$

As shown in section 1, we know

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

Using this we get,

$$\boldsymbol{E}_\sigma = \frac{2\mu}{\left(1-\frac{\lambda}{2(\lambda+\mu)}\right)}\begin{bmatrix}1 & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{1}{2}\left(1-\frac{\lambda}{2(\lambda+\mu)}\right)\end{bmatrix}$$

Simplifying, we get,

$$\boldsymbol{E}_\sigma = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\begin{bmatrix}1 & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{\lambda+2\mu}{4\lambda+4\mu}\end{bmatrix}$$

Hence,

$$\boxed{\boldsymbol{E}_\sigma = \begin{bmatrix}\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & 0\\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0\\ 0 & 0 & \mu\end{bmatrix}}$$

Similarly, we know the elastic matrix for plane strain condition $\boldsymbol{E}_\epsilon$ is given as,

$$\boldsymbol{E}_\epsilon = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Given that,

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

we have,

$$\boldsymbol{E}_\epsilon = \frac{\lambda(1+\nu)(1-2\nu)(1-\nu)}{\nu(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

which on simplifying results in,

$$\boldsymbol{E}_\epsilon = \frac{\lambda(1-\nu)}{\nu}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Further simplification leads us to,

$$\boldsymbol{E}_\epsilon = \begin{bmatrix} \frac{\lambda(1-\nu)}{\nu} & \lambda & 0 \\ \lambda & \frac{\lambda(1-\nu)}{\nu} & 0 \\ 0 & 0 & \frac{\lambda(1-2\nu)}{2\nu} \end{bmatrix}$$

$$\boldsymbol{E}_\epsilon = \begin{bmatrix} \lambda(\frac{1}{\nu}-1) & \lambda & 0 \\ \lambda & \lambda(\frac{1}{\nu}-1) & 0 \\ 0 & 0 & \lambda(\frac{1}{2\nu}-1) \end{bmatrix}$$

Again, using the expression for ν, we get,

$$\boxed{\boldsymbol{E}_\epsilon = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}$$

3. Split the stress-strain matrix E for plane strain as

$$\boldsymbol{E} = \boldsymbol{E}_\lambda + \boldsymbol{E}_\mu$$

in which $\boldsymbol{E}_\mu$ and $\boldsymbol{E}_\lambda$ contain only μ and λ, respectively.

This is the Lamé $\{\lambda, \mu\}$ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

Solution: In the previous section, the the stress-strain matrix E for plane strain was deduced as,

$$\boldsymbol{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

To perform Lamé splitting for the plane strain condition, we split the elastic matrix $\boldsymbol{E}$ into $\boldsymbol{E}_\mu$ and $\boldsymbol{E}_\lambda$ containing only μ and λ respectively, as

$$\boldsymbol{E} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

where,

$$\boxed{\boldsymbol{E}_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{E}_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}$$

4. Express $\boldsymbol{E}_\lambda$ and $\boldsymbol{E}_\mu$ also in terms of E and ν.

Solution: Using the expressions given for λ and μ as,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}$$

we express $\boldsymbol{E}_\lambda$ and $\boldsymbol{E}_\mu$ as,

$$\boxed{\boldsymbol{E}_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{E}_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Assignment 3.2

On "The 3-node Plane Stress Triangle":

Consider a plane triangular domain of thickness h, with horizontal and vertical edges of length a. Let us consider for simplicity $a = 1$, $h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two discrete structural models are considered as depicted in the figure:

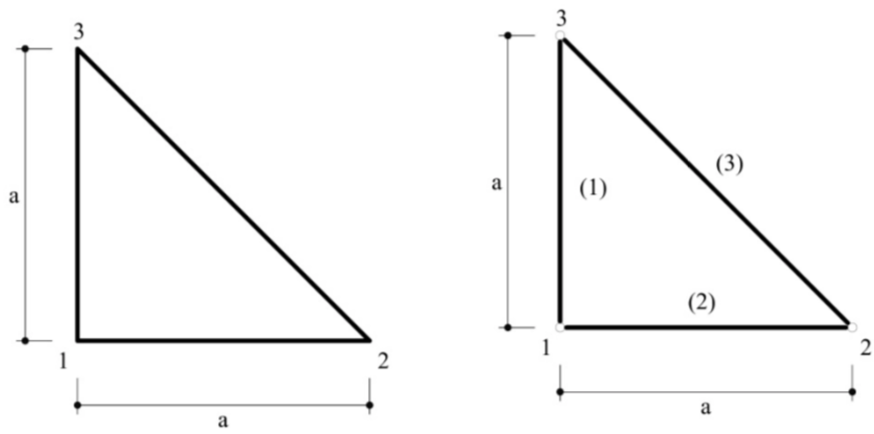

Figure 1

a) A plane linear Turner triangle with the same dimensions.

b) A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

1. Calculate the stiffness matrices $\boldsymbol{K}_{tri}$ and $\boldsymbol{K}_{bar}$ for both discrete models.

Solution: The element stiffness matrix for a linear triangle element is given as,

$$\boldsymbol{K}_{tri} = \boldsymbol{K}^e = \int_{\Omega^e} h\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}\,d\Omega$$

Given that the plane triangular domain is of constant thickness $h = 1$, we get the expression for element stiffness matrix as,

$$\boldsymbol{K}_{tri} = hA\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}$$

For plane stress case, we know,

$$\boldsymbol{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

It is also given that initially, ν is set to zero, which gives us,

$$\boldsymbol{E} = E\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{bmatrix}$$

Also, the matrix $\boldsymbol{B}$ is given as,

$$\boldsymbol{B} = \frac{1}{2A}\begin{bmatrix}y_{23} & 0 & y_{31} & 0 & y_{12} & 0\\0 & x_{32} & 0 & x_{13} & 0 & x_{21}\\x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}\end{bmatrix}$$

where,

$$y_{12} = y_1 - y_2 = 0; \quad y_{23} = y_2 - y_3 = -1; \quad y_{31} = y_3 - y_1 = 1$$

$$x_{13} = x_1 - x_3 = 0; \quad x_{32} = x_3 - x_2 = -1; \quad x_{21} = x_2 - x_1 = 1$$

Thus we find the element stiffness matrix as,

$$\boldsymbol{K}_{tri} = \frac{hA}{4A^2}\begin{bmatrix}y_{23} & 0 & x_{32}\\0 & x_{32} & y_{23}\\y_{31} & 0 & x_{13}\\0 & x_{13} & y_{31}\\y_{12} & 0 & x_{21}\\0 & x_{21} & y_{12}\end{bmatrix}\ \boldsymbol{E}\ \begin{bmatrix}y_{23} & 0 & y_{31} & 0 & y_{12} & 0\\0 & x_{32} & 0 & x_{13} & 0 & x_{21}\\x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12}\end{bmatrix}$$

Using the values derived above and noting that $A = 1/2$ is the area of the triangle, we get,

$$\boldsymbol{K}_{tri} = \frac{E}{2}\begin{bmatrix}-1 & 0 & -1\\0 & -1 & -1\\1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0 & 1\\-1 & -1 & 0 & 1 & 1 & 0\end{bmatrix}$$

Hence,

$$\boxed{\boldsymbol{K}_{tri} = \frac{E}{4}\begin{bmatrix}3 & 1 & -2 & -1 & -1 & 0\\1 & 3 & 0 & -1 & -1 & -2\\-2 & 0 & 2 & 0 & 0 & 0\\-1 & -1 & 0 & 1 & 1 & 0\\-1 & -1 & 0 & 1 & 1 & 0\\0 & -2 & 0 & 0 & 0 & 2\end{bmatrix}}$$

Now, in order to find the stiffness matrix $\boldsymbol{K}_{bar}$, we need to get the element stiffness matrix of each bar element given as,

$$\boldsymbol{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

where $s = \sin\alpha$, $c = \cos\alpha$ and α is the angle formed by the bar element with the global x-coordinate measured in the counter-clockwise direction.

Now, for element 1, $\alpha = \pi/2 \implies c = 0$ and $s = 1$

$$\boldsymbol{K}^{(1)} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Similarly, for element 2, $\alpha = 0 \implies c = 1$ and $s = 0$

$$\boldsymbol{K}^{(2)} = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And, for element 3, $\alpha = 3\pi/4 \implies c = -1/\sqrt{2}$ and $s = 1/\sqrt{2}$

$$\boldsymbol{K}^{(3)} = \frac{EA_3}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We note that the cross sections for the bar elements are given as $A_1 = A_2$ and assemble the above elemental matrices to give us the global stiffness matrix as,

$$\boldsymbol{K}_{bar} = \frac{E}{2\sqrt{2}} \begin{bmatrix} 2A_1\sqrt{2} & 0 & -2A_1\sqrt{2} & 0 & 0 & 0 \\ 0 & 2A_1\sqrt{2} & 0 & 0 & 0 & -2A_1\sqrt{2} \\ -2A_1\sqrt{2} & 0 & 2A_1\sqrt{2}+A_3 & -A_3 & -A_3 & A_3 \\ 0 & 0 & -A_3 & A_3 & A_3 & -A_3 \\ 0 & 0 & -A_3 & A_3 & A_3 & -A_3 \\ 0 & -2A_1\sqrt{2} & A_3 & -A_3 & -A_3 & 2A_1\sqrt{2}+A_3 \end{bmatrix}$$

2. Is there any set of values for the cross sections $A_1 = A_2$ and A_3 to make both stiffness matrix equivalent: $\boldsymbol{K}_{tri} = \boldsymbol{K}_{bar}$? If not, which are the values that make them more similar?

Solution: The stiffness matrices $\boldsymbol{K}_{tri}$ and $\boldsymbol{K}_{bar}$ derived for both discrete models are given as,

$$\boldsymbol{K}_{tri} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{K}_{bar} = \frac{E}{4}\begin{bmatrix} 4A_1 & 0 & -4A_1 & 0 & 0 & 0 \\ 0 & 4A_1 & 0 & 0 & 0 & -4A_1 \\ -4A_1 & 0 & 4A_1 + A_3\sqrt{2} & -A_3\sqrt{2} & -A_3\sqrt{2} & A_3\sqrt{2} \\ 0 & 0 & -A_3\sqrt{2} & A_3\sqrt{2} & A_3\sqrt{2} & -A_3\sqrt{2} \\ 0 & 0 & -A_3\sqrt{2} & A_3\sqrt{2} & A_3\sqrt{2} & -A_3\sqrt{2} \\ 0 & -4A_1 & A_3\sqrt{2} & -A_3\sqrt{2} & -A_3\sqrt{2} & 4A_1 + A_3\sqrt{2} \end{bmatrix}$$

Comparing the above matrices, we note that for cross sections $A_1 = A_2$ and A_3, there are no set of values to make both stiffness matrix equivalent. This is inferred by noticing the uneven distribution of zeros in the off-diagonal terms of both matrices. Therefore it is not possible to make $\boldsymbol{K}_{tri} = \boldsymbol{K}_{bar}$. But it is possible to make them more similar by attempting to equate the non-zero stiffness coefficients in both matrix. Firstly we see that the diagonal of both matrices has non-zero entries.

Therefore, if we equate $\boldsymbol{K}_{tri_{11}} = \boldsymbol{K}_{bar_{11}}$ to get $4A_1 = 3 \implies A_1 = 3/4$. Similarly, equating $\boldsymbol{K}_{tri_{44}} = \boldsymbol{K}_{bar_{44}}$, we get $A_3 = 1/\sqrt{2}$.

By assuming these values, we can make 14 stiffness coefficients out of the 6×6 matrix similar in both matrices, which are highlighted below. The values highlighted with green are exactly matching in both matrices whereas the ones marked with blue are similar. All other stiffness coefficients can not be matched.

$$\boldsymbol{K}_{tri} = \frac{E}{4}\begin{pmatrix} \boxed{3} & 1 & \boxed{-2} & -1 & -1 & \boxed{0} \\ 1 & \boxed{3} & \boxed{0} & -1 & -1 & \boxed{-2} \\ \boxed{-2} & \boxed{0} & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & \boxed{1} & \boxed{1} & 0 \\ -1 & -1 & 0 & \boxed{1} & \boxed{1} & 0 \\ \boxed{0} & \boxed{-2} & 0 & 0 & 0 & 2 \end{pmatrix} \quad \boldsymbol{K}_{bar} = \frac{E}{4}\begin{pmatrix} \boxed{3} & 0 & \boxed{-3} & 0 & 0 & \boxed{0} \\ 0 & \boxed{3} & \boxed{0} & 0 & 0 & \boxed{-3} \\ \boxed{-3} & \boxed{0} & 4 & -1 & -1 & 1 \\ 0 & 0 & -1 & \boxed{1} & \boxed{1} & -1 \\ 0 & 0 & -1 & \boxed{1} & \boxed{1} & -1 \\ \boxed{0} & \boxed{-3} & 1 & -1 & -1 & 4 \end{pmatrix}$$

3. Why these two stiffness matrices are not equal? Find a physical explanation.

Solution: It is seen that it's not possible to make the two stiffness matrices equal for any set of values of the cross sections of the bar element. This is because the two elements compared look similar - set of three bar elements in a triangular domain and plane linear turner triangle with the same dimensions, but behave differently in presence of applied force or displacement. It is because the material distribution is even throughout the area of the triangular element whereas a set of bars only consider the material concentrating on the bar elements and not in the triangular domain formed as shown in Figure 2. Another fact to consider is the presence of thickness of the triangular element, which is not incorporated exactly by the set of bar elements. Therefore it is unfair to try to match the stiffness matrices for both cases.

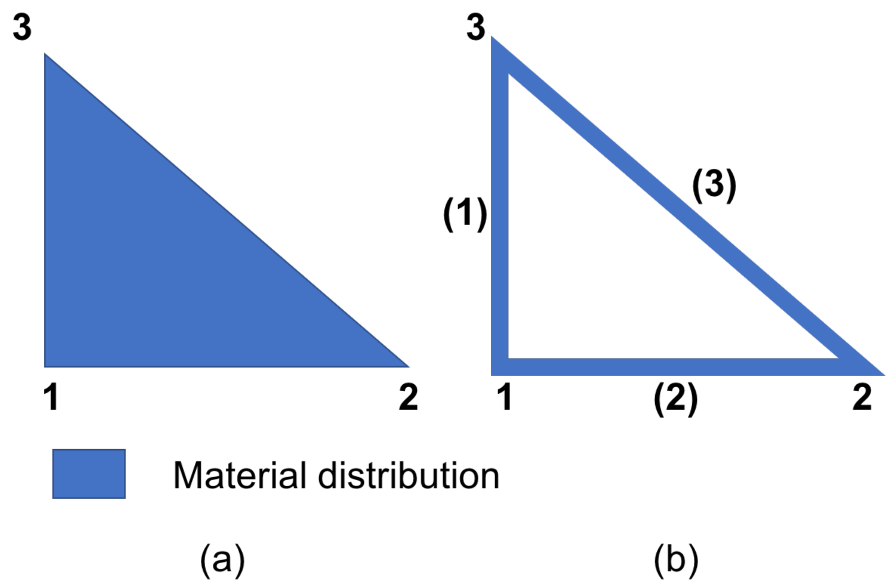

Figure 2: Material distribution in (a) triangular element and (b) set of bars

4. Consider now $\nu \neq 0$ and extract some conclusions.

Solution: For $\nu \neq 0$, the matrix $\boldsymbol{E}$ changes in the plane stress case as,

$$\boldsymbol{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

We notice that the matrix is still symmetric but not a diagonal matrix as in the earlier case. This physically means that now the Poisson's effect is considered for the element i.e. the stress accounts for shear effects also. Using the new matrix $\boldsymbol{E}$, we get the stiffness matrix as,

$$\boldsymbol{K}_{tri} = \frac{E}{4(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & -1+\nu & -1+\nu & -2\nu \\ 1+\nu & 3-\nu & -2\nu & -1+\nu & -1+\nu & -2 \\ -2 & -2\nu & 2 & 0 & 0 & 2\nu \\ -1+\nu & -1+\nu & 0 & 1-\nu & 1-\nu & 0 \\ -1+\nu & -1+\nu & 0 & 1-\nu & 1-\nu & 0 \\ -2\nu & -2 & 2\nu & 0 & 0 & 2 \end{bmatrix}$$

The above matrix clearly shows the effect of including the poisson's ratio in the stiffness calculation. We notice that almost all the stiffness coefficients have an impact of this inclusion. The presence of the term ν in most of the coefficients means that the stress in one direction has an effect in all elements.