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Assignment 3.1

Plane stress
$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{pmatrix}$$

Plane strain
$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{pmatrix}$$

Plane stress

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\nu}{1-\nu^2}\varepsilon_{yy}$$

$$\sigma_{yy} = \frac{E\nu}{1-\nu^2}\varepsilon_{xx} + \frac{E}{1-\nu^2}\varepsilon_{yy}$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

Plane strain

$$\sigma_{xy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{yy}$$

$$\sigma_{yy} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{yy}$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

a)

$$\frac{E^*}{1-\nu^{*2}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad ① \qquad \frac{E^*\nu^*}{1-\nu^{*2}} = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad ②$$

② ÷ ①

$$\boxed{\nu^* = \frac{\nu}{1-\nu}}$$

$$\frac{E^*}{1+\nu^*} = \frac{E}{1+\nu}$$

$$E^* = \frac{E}{1+\nu}(1+\nu^*)$$

$$E^* = \frac{E}{1+\nu}\left(1+\frac{\nu}{1-\nu}\right)$$

$$\boxed{E^* = \frac{E}{1-\nu^2}}$$

$$\frac{E^*}{1-\nu^{*2}}\begin{bmatrix}1 & \nu^* & 0\\ \nu^* & 1 & 0\\ 0 & 0 & \frac{1-\nu^*}{2}\end{bmatrix}$$

replace E^*, ν^* $\rightarrow$

$$\left(\frac{E}{1-\nu^2}\right)\left(\frac{1}{1-\frac{\nu^2}{(1-\nu)^2}}\right)\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-\frac{\nu}{1-\nu}}{2}\end{bmatrix} = \frac{E}{1-\nu^2}\,\frac{(1-\nu)^2}{1-2\nu}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}$$

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

b) $$\frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2} \quad ③ \qquad \frac{E^*\nu^*}{(1+\nu^*)(1-2\nu^*)} = \frac{E\nu}{1-\nu^2} \quad ④$$

④ ÷ ③

$$\frac{\nu^*}{1-\nu^*} = \nu$$

$$\nu^* = \nu - \nu\nu^*$$

$$\boxed{\nu^* = \frac{\nu}{1+\nu}}$$

$$\frac{E^*}{1+\nu^*} = \frac{E}{1+\nu}$$

$$\frac{E^*}{1+\frac{\nu}{1+\nu}} = \frac{E}{1+\nu}$$

$$E^* = \frac{E}{1+\nu}\left(\frac{1+\nu+\nu}{1+\nu}\right)$$

$$\boxed{E^* = \frac{E(1+2\nu)}{(1+\nu)^2}}$$

$$\frac{E^*}{(1+\nu^*)(1-2\nu^*)}\begin{pmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & \frac{1-2\nu^*}{2} \end{pmatrix} \xrightarrow[E^*,\nu^*]{\text{Replace}} \frac{E(1+2\nu)}{(1+\nu)^2}\left(\frac{1}{\left(1+\frac{\nu}{1+\nu}\right)\left(1-\frac{2\nu}{1+\nu}\right)}\right)\begin{bmatrix} 1-\frac{\nu}{1+\nu} & \frac{\nu}{1+\nu} & 0 \\ \frac{\nu}{1+\nu} & 1-\frac{\nu}{1+\nu} & 0 \\ 0 & 0 & \frac{1-\frac{2\nu}{1+\nu}}{2} \end{bmatrix} =$$

$$= \frac{E\cancel{(1+2\nu)}}{\cancel{(1+\nu)^2}}\left(\frac{\cancel{(1+\nu)^2}}{\cancel{(1+2\nu)}(1-\nu)}\right)\begin{bmatrix} \frac{1}{1+\nu} & \frac{\nu}{1+\nu} & 0 \\ \frac{\nu}{1+\nu} & \frac{1}{1+\nu} & 0 \\ 0 & 0 & \frac{1-\nu}{2(1+\nu)} \end{bmatrix}$$

$$= \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad //$$

Assignment 3.2

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

a)

$$\lambda = 2\frac{E}{2(1+\nu)}\frac{\nu}{1-2\nu} = 2\mu\frac{\nu}{1-2\nu}$$

$$\frac{\lambda}{2\mu}(1-2\nu) = \nu$$

$$\nu\left(1+\frac{\lambda}{\mu}\right) = \frac{\lambda}{2\mu}$$

$$\nu = \frac{\lambda\cancel{\mu}}{2(\lambda+\mu)\cancel{\mu}}$$

$$\boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}}$$

$$E = 2\mu(1+\nu)$$

$$E = 2\mu\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)$$

$$E = 2\mu\left(\frac{2\lambda+2\mu+\lambda}{2(\lambda+\mu)}\right)$$

$$\boxed{E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}$$

b) Plane stress

$$\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\cdot\frac{1}{1-\frac{\lambda^2}{4(\lambda+\mu)^2}}\begin{bmatrix}1 & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{2}\end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\frac{4(\lambda+\mu)^2}{4\lambda^2+8\lambda\mu+4\mu^2-\lambda^2}\begin{bmatrix}1 & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{2}\end{bmatrix} = \frac{4\mu(3\lambda+2\mu)(\lambda+\mu)}{4\mu^2+8\mu\lambda+3\lambda^2}\begin{bmatrix}1 & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)}\end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{(3\lambda+2\mu)(\lambda+2\mu)}\begin{bmatrix}4(\lambda+\mu) & 2\lambda & 0\\ 2\lambda & 4(\lambda+\mu) & 0\\ 0 & 0 & \lambda+2\mu\end{bmatrix} = \frac{\mu}{\lambda+2\mu}\begin{bmatrix}4(\lambda+\mu) & 2\lambda & 0\\ 2\lambda & 4(\lambda+\mu) & 0\\ 0 & 0 & \lambda+2\mu\end{bmatrix}$$

//

b) Plane strain

$$\frac{E\cdot\nu}{\nu(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix} = \frac{\lambda}{\nu}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix} = \lambda\begin{bmatrix}\frac{1}{\nu}-1 & 1 & 0\\ 1 & \frac{1}{\nu}-1 & 0\\ 0 & 0 & \frac{1}{2\nu}-1\end{bmatrix}$$

$$= \lambda\begin{bmatrix}\frac{2(\lambda+\mu)}{\lambda}-1 & 1 & 0\\ 1 & \frac{2(\lambda+\mu)}{\lambda}-1 & 0\\ 0 & 0 & \frac{\lambda+\mu}{\lambda}-1\end{bmatrix} = \lambda\begin{bmatrix}\frac{\lambda+2\mu}{\lambda} & 1 & 0\\ 1 & \frac{\lambda+2\mu}{\lambda} & 0\\ 0 & 0 & \frac{\mu}{\lambda}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

//

c) Split the stress-strain matrix E of plane strain as

$$E = E_\lambda + E_\mu$$

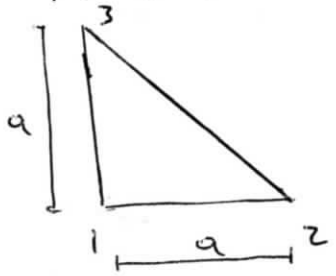
$$E = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$E_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

d) $$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

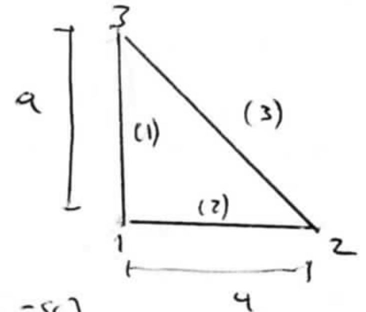
Assignment 3.3

Plane linear corner triangle



A set of three bar elements

$A_1 = A_2 \qquad A_3$



$a = 1$
$h = 1$
$\nu = 0$

$$K^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Bars elements

$$K^1 = \frac{EA_1}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad K^2 = \frac{EA_2}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K^3 = \frac{EA_3}{\sqrt{2}}\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$K_{bar} = E\begin{pmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_2 & 0 & A_2 + \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{pmatrix} = E\begin{pmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ 0 & -A & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & A + \frac{A'}{2\sqrt{2}} \end{pmatrix} \; //$$

Plane linear triangle

$$K^e = \frac{h}{4A}\begin{pmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{pmatrix} \cdot E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix}$$

$$k = \frac{E}{2}\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K_{tr} = \frac{E}{2}\begin{pmatrix} -1 & 0 & -1/2 \\ 0 & -1 & -1/2 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} = \frac{E}{2}\begin{pmatrix} 3/2 & 1/2 & -1 & -1/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 & -1/2 & -1/2 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \; //$$

b) The stiffness matrix cannot be equivalente $K_{bar} \neq K_{triangle}$

c) The barelements just transmit axial forces

d) $$K^e = \frac{E}{2(1-\nu^2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e = \frac{E}{2(1-\nu^2)} \begin{bmatrix} -1 & -\nu & -\frac{1-\nu}{2} \\ -\nu & -1 & -\frac{1-\nu}{2} \\ 1 & \nu & 0 \\ 0 & 0 & \frac{1-\nu}{2} \\ 0 & 0 & \frac{1-\nu}{2} \\ \nu & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e = \frac{E}{2(1-\nu^2)} \begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

- The matrix is still symmetric
- Some components change to non-zero values. However, some components are still zero