Computational Structural Mechanics and Dynamics **February 26, 2019**
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Assignment 3

3.1 Suppose that the structural material is isotropic, with elastic modulus E and Poissons ratio . The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are

$$\text{Plane stress: } \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix} \tag{1}$$

$$\text{Plane strain: } \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix} \tag{2}$$

a)

$$\text{Plane stress E matrix: } \begin{bmatrix}\dfrac{E}{(1-\nu^2)} & \dfrac{E\nu}{(1-\nu^2)} & 0\\ \dfrac{E\nu}{(1-\nu^2)} & \dfrac{E}{(1-\nu^2)} & 0\\ 0 & 0 & \dfrac{E}{2(1+\nu)}\end{bmatrix} \tag{3}$$

Replacement rule:

$$E^* = \frac{E}{(1-\nu^2)} \qquad \nu^* = \frac{\nu}{-1+\nu} \tag{4}$$

Then,

$$\text{Plane strain E matrix: } \frac{E^*}{1-(\nu^*)^2}\begin{bmatrix}1 & \nu^* & 0\\ \nu^* & 1 & 0\\ 0 & 0 & \dfrac{1-\nu^*}{2}\end{bmatrix} \tag{5}$$

b)

$$\text{Plane strain E matrix: } \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{E\nu}{(1-2\nu)(1+\nu)} & 0 \\ \frac{E\nu}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \tag{6}$$

Replacement rule:

$$E^* = \frac{E(1+2\nu)}{(1+\nu^2)}$$
$$\nu^* = \frac{\nu}{1+\nu} \tag{7}$$

$$\text{Plane stress E matrix:} \frac{E^*}{(1+\nu^*)(1-2\nu^*)} \begin{bmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & \frac{1-2\nu^*}{2} \end{bmatrix} \tag{8}$$

3.2 In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called Lam constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)} \tag{9}$$

a) **Find the inverse relations for E,ν in terms of λ, μ.**

First we are going to simplify the expression μ/λ:

$$\frac{\mu}{\lambda} = \frac{\frac{E}{2(1+\nu)}}{\frac{E\nu}{(1+\nu)(1-2\nu)}}$$
$$\Downarrow$$
$$\frac{\mu}{\lambda} = \frac{1-2\nu}{2\nu}$$

$$2\nu\mu = (1-2\nu)\lambda$$

$$2\nu\mu = \lambda - 2\nu\lambda$$

$$2\nu\mu + 2\nu\lambda = \lambda$$

$$\boxed{\nu = \frac{1}{2}\frac{\lambda}{\lambda+\mu}} \tag{10}$$

If we now substitute ν in the second expression of (9):

$$\mu = \frac{E}{2\left(1+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\right)}$$

$$\mu = \frac{E}{\frac{2(\lambda+\mu)+\lambda}{\lambda+\mu}}$$

$$\mu = \frac{E}{\frac{3\lambda+2\mu}{\lambda+\mu}}$$

$$\boxed{E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}} \tag{11}$$

b) **Express the elastic matrix for plane stress and plane strain cases in terms of λ,μ.**

For plane stress:

$$\text{Plane stress:} = E = \begin{bmatrix} \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & 0 \\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

If we use a a modified Lamé constant :

$$\bar{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu} = \frac{E\nu}{1-\nu^2}$$

We have:

$$= \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \bar{\lambda} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{E\nu}{1+\nu^2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_\mu + E_\lambda$$

For plane strain:

$$\text{Plane strain:} = E = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_\mu + E_\lambda$$

c) **Split the stress-strain matrix E of plane strain as:**

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda$$

Done in the previous section.

d) **Express E_μ and E_λ also in terms of E and ν.**

$$E_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.3 Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Lets consider for simplicity a = h = 1. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

a) **Calculate the stiffness matrix K^e for both models.**

- Three Bars:
  The elemental global matrix for truss elements is:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \tag{12}$$

  Now we calculate K for each element:

  - Bar 1

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x3} \\ f_{y3} \end{bmatrix} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{13}$$

  - Bar 2

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} \tag{14}$$

  - Bar 3

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \frac{EA_3\sqrt{2}}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{15}$$

  Then the global stiffness matrix of the system is:

$$\mathbf{K}^e = E \begin{bmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & A_2 + \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} & \frac{A_3\sqrt{2}}{4} \\ & & & \frac{A_3\sqrt{2}}{4} & \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} \\ & SYM & & & \frac{A_3\sqrt{2}}{4} & -\frac{A_3\sqrt{2}}{4} \\ & & & & & A_1 + \frac{A_3\sqrt{2}}{4} \end{bmatrix} \tag{16}$$

- Turner Triangle:

  The element stiffness matrix of the "Turner Triangle" is defined as:

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B}d\Omega \tag{17}$$

As $\mathbf{B}, \mathbf{E}$ and A are constants, and also $h = 1$ and $a = 1$, the expression is integrated as:

$$\mathbf{K}^e = \frac{1}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{18}$$

where $x_{jk} = x_j - x_k$ and $y_{jk} = y_j - y_k$.

Node	X	Y
1	0.0	0.0
2	1.0	0.0
3	0.0	1.0

Table 1: Nodal coordinates.

Substituting the coordinates into the above equation:

$$\mathbf{K}^e = \frac{1}{4A}\begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} -1.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ -1.0 & -1.0 & 0.0 & 1.0 & 1.0 & 0.0 \end{bmatrix}$$

Considering the plane stress constitutive matrix:

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{19}$$

considering initially $\nu = 0$ and substituting the corresponding values:

$$\mathbf{K}^e = \frac{E}{8A}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & SYM & & & 1 & 0 \\ & & & & & 2 \end{bmatrix} \tag{20}$$

b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $\mathbf{K_{bar}} = \mathbf{K_{triangle}}$? If not, which are these values to make them as similar as possible?

Taking into account the cross sections equal for bar 1 and bar 2, the stiffness matrix of the first system is modified as next:

$$\mathbf{K}^e = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + \frac{A'\sqrt{2}}{4} & -\frac{A'\sqrt{2}}{4} & -\frac{A'\sqrt{2}}{4} & \frac{A'\sqrt{2}}{4} \\ & & & \frac{A'\sqrt{2}}{4} & \frac{A'\sqrt{2}}{4} & -\frac{A'\sqrt{2}}{4} \\ & SYM & & & \frac{A'\sqrt{2}}{4} & -\frac{A'\sqrt{2}}{4} \\ & & & & & A + \frac{A'\sqrt{2}}{4} \end{bmatrix} \tag{21}$$

The first thing that we observed is that it is not possible way to obtaine the same matrix (some of the values are equal to zero in the stiffness matrix of the truss system) but we can try to make it as similar as possible starting with the diagonal elements:

Case 1:

$$K_{11} \rightarrow EA = \frac{3E}{8A^*} \rightarrow A = \frac{3}{4}$$

$$K_{33} \rightarrow E\left(A + \frac{A'\sqrt{2}}{4}\right) = \frac{2E}{8A^*} \rightarrow A' = -\frac{\sqrt{2}}{2}$$

Case 2:

$$K_{11} \rightarrow A = \frac{3}{4}$$

$$K_{44} \rightarrow E\left(\frac{A'\sqrt{2}}{4}\right) = \frac{E}{8A^*} \rightarrow A' = \frac{\sqrt{2}}{2}$$

Case 3:

$$K_{44} \rightarrow E\left(\frac{A'\sqrt{2}}{4}\right) = \frac{E}{8A^*} \rightarrow A' = \frac{\sqrt{2}}{2}$$

$$K_{33} \rightarrow E\left(A + \frac{A'\sqrt{2}}{4}\right) = \frac{2E}{8A^*} \rightarrow A = \frac{1}{4}$$

In the first case one of the areas is negative, which is why we rejected this solution. For cases 2 and 3, the matrices are formed as follows:

Case 2:

$$\mathbf{K}_{bar2} = \frac{E}{4}\begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & -3 \\ & & 4 & -1 & -1 & 1 \\ & & & 1 & 1 & -1 \\ & SYM & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}$$

Case 3:

$$\mathbf{K}_{bar3} = \frac{E}{4}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1 \\ & & 2 & -1 & -1 & 1 \\ & & & 1 & 1 & -1 \\ & SYM & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}$$

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

The bar is obtained by considering a linear element from node to node. The truss system are three bars in shape of a triangule empty in the interior, the Turner's triangle is different then, we are considering a continuous element with the shape of a triangle.

Even though both analysis consider linear numerical approximation, they can not be completely taken as equal.

What happens in going from 1D to 2D? New effects emerge, notably shear energy and inplane bending.

d) Solve question a) considering $\nu \neq 0$ and extract some conclusions

$$\mathbf{K}^e = \frac{E}{8A(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & \nu-1 & \nu-1 & -2\nu \\ & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ & & 2 & 0 & 0 & 2\nu \\ & & & 1-\nu & 1-\nu & 0 \\ & SYM & & & 1-\nu & 0 \\ & & & & & 2 \end{bmatrix} \tag{22}$$