Assignment 3.1

1.

$x_1 = 0,\ x_2 = 3,\ x_3 = 2$
$y_1 = 0,\ y_2 = 1,\ y_3 = 2 \rightarrow$ Area= 2

$$\boldsymbol{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1$$

The element stiffness matrix is given by the general formula.

$$\mathbf{K}^e = \int_{\Omega^e} h\, \mathbf{B}^T \mathbf{E} \mathbf{B}\, d\Omega^e,$$

where Ωe is the triangle domain, and h the plate thickness that appears in the plane stress problem.
Since **B** and **E** are constant, they can be taken out of the integral:

$$\mathbf{K}^e = \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^e} h\, d\Omega$$

If h is uniform over the element the remaining integral is simply hA, and we obtain the closed form

$$\mathbf{K}^e = A\,h\,\mathbf{B}^T\mathbf{E}\,\mathbf{B} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}.$$

$$K^e = 1/8 \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} \frac{75}{4} & \frac{75}{8} & \frac{-25}{2} & \frac{-25}{4} & \frac{-25}{4} & \frac{-25}{8} \\ \frac{75}{8} & \frac{75}{4} & \frac{25}{4} & \frac{25}{2} & \frac{-125}{8} & \frac{-125}{4} \\ \frac{-25}{2} & \frac{25}{4} & 75 & \frac{-75}{2} & \frac{-125}{2} & \frac{125}{4} \\ \frac{-25}{4} & \frac{25}{2} & \frac{-75}{2} & 75 & \frac{175}{4} & \frac{-175}{2} \\ \frac{-25}{4} & \frac{-125}{8} & \frac{-125}{2} & \frac{175}{4} & \frac{275}{4} & \frac{-225}{8} \\ \frac{-25}{8} & \frac{-125}{4} & \frac{125}{4} & \frac{-175}{2} & \frac{-225}{8} & \frac{475}{4} \end{pmatrix}$$

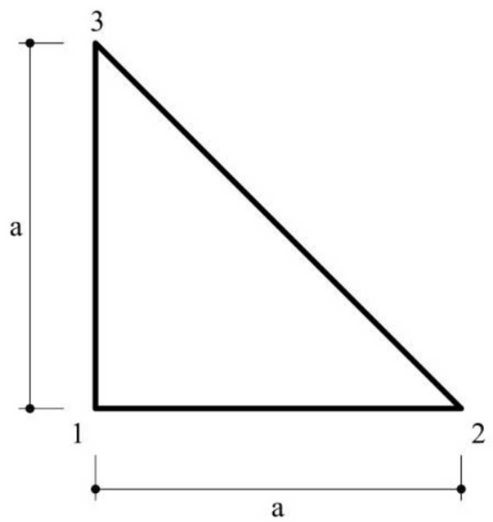
2.

The reason why the sum of the entries in each row and column is zero is because they are linearly dependent, so the matrix is singular. It does make sense because we do not have any boundary conditions
We have to notice that the linear shape function takes values of unity on the current node and zero at the others.

Assignment 3.2

a)

For plane linear Turner triangle



$x_1 = 0$, $x_2 = a$, $x_3 = 0$
$y_1 = 0$, $y_2 = 0$, $y_3 = a$

For simplicity h=a=1 → A=1/2

For isotropic material characterized by Young modulus (E) and Poison coefficient (ν) the constitutive relation between stresses and strains is

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E}\,\mathbf{e},$$

where Ei j are plane stress elastic moduli. The constitutive matrix E will be assumed to be constant over the element. Because the strains are constant, so are the stresses.

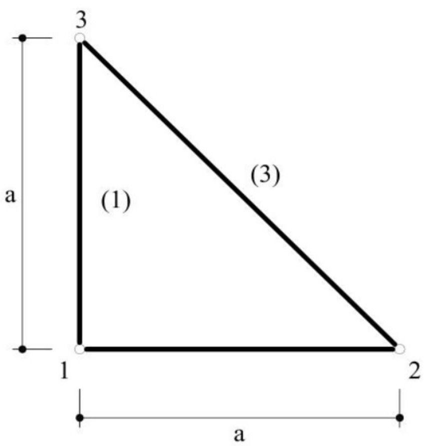
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix} \quad \xrightarrow{\upsilon=0} \quad E= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$K^e = 1/2 \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$K^e = \begin{pmatrix} 0.75 & 0.25 & -0.50 & -0.25 & -0.25 & 0.00 \\ 0.25 & 0.75 & 0.00 & -0.25 & -0.25 & -0.50 \\ -0.50 & 0.00 & 0.50 & 0.00 & 0.00 & 0.00 \\ -0.25 & -0.25 & 0.00 & 0.25 & 0.25 & 0.00 \\ -0.25 & -0.25 & 0.00 & 0.25 & 0.25 & 0.00 \\ 0.00 & -0.50 & 0.00 & 0.00 & 0.00 & 0.50 \end{pmatrix}$$

<u>For the three bar element</u>

Using the direct stiffness method



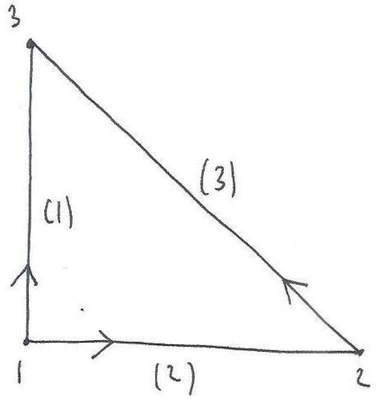
Element	i	j	Area	L	ϑ	EA/L	cos(ϑ)	sin(ϑ)	cos(ϑ)sin(ϑ)
1	1	3	A	1	π/2	EA	0	1	0
2	1	2	A	1	0	EA	1	0	0
3	2	3	A'	√2	3π/4	EA'	-√2/2	√2/2	-0,5

Global stiffness matrix by element

$$K^{(1)} = EA \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$K^{(2)} = EA \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^{(3)} = \frac{EA'}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$



$$K^e = \begin{pmatrix} K_{aa}^1 + K_{aa}^2 & K_{ab}^2 & K_{ab}^1 \\ & K_{aa}^3 + K_{bb}^2 & K_{ab}^3 \\ \text{Sym} & & K_{bb}^1 + K_{bb}^3 \end{pmatrix}$$

$$K^e = \begin{pmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + \frac{A'}{2\sqrt{2}} & \frac{-A'}{2\sqrt{2}} & \frac{-A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ & & & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{-A'}{2\sqrt{2}} \\ & \text{Sym} & & & \frac{A'}{2\sqrt{2}} & \frac{-A'}{2\sqrt{2}} \\ & & & & & A + \frac{A'}{2\sqrt{2}} \end{pmatrix}$$

b) and c)

The zero entries that we have in the first stiffness matrix does not match with the values in the second so it is not possible to obtain same the matrix
However, if we were to put $A = 0.25$, $A' = \sqrt{2}/2$ then we would obtain the same last 4 values of the diagonal.

The two matrix represent the global stiffness of different structural problems. The bar triangle behaves like a truss, which means that the elements can only hold axial stresses and the boundary conditions are applied at the nodes.
On the other hand the Turner triangle has stiffness in any direction, not only in the longitudinal way of the bar and you can apply distributed loads along its perimeter.

d)

Poisson's ratio describes the expansion of a material in directions perpendicular to the direction of compression.
If the Poisson ratio is zero, the stresses are given by a diagonal matrix, meaning that only a displacement in the direction of the axis will cause stresses in that axis.
If the Poisson ratio is different than zero, the whole structure will be more stiff because the transversal strain appears in the material.
Making the calculations the matrix results like this

$$K^e = \frac{1}{2}\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix} =$$

$$= \frac{E}{2(1-\nu^2)}\begin{pmatrix} 1{,}5-\nu/2 & 0{,}5+\nu/2 & -1 & \nu/2-0{,}5 & \nu/2-0{,}5 & -\nu \\ \nu/2+0{,}5 & 1{,}5-\nu/2 & -\nu & \nu/2-0{,}5 & \nu/2-0{,}5 & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \nu/2-0{,}5 & \nu/2-0{,}5 & 0 & 0{,}5-\nu/2 & 0{,}5-\nu/2 & 0 \\ \nu/2-0{,}5 & \nu/2-0{,}5 & 0 & 0{,}5-\nu/2 & 0{,}5-\nu/2 & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{pmatrix}$$