Universitat Politècnica de Catalunya
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Assingment 3 -The Plane Stress Problem and the 3-Node Plane Stress Triangle

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3.1

1 - E and ν in terms of λ and μ

Given the equations for the Lamé constants λ and μ:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{1}$$

$$\mu = \frac{E}{2(1+\nu)} \tag{2}$$

Solving both equations, (1 and 2), for E we have.

$$\frac{\cancel{(1+\nu)}(1-2\nu)}{\nu}\lambda = E = 2\cancel{(1+\nu)}\mu$$

$$(1-2\nu)\lambda = 2\mu\nu$$

$$2\mu\nu + 2\lambda\nu = \lambda$$

Which leads to the definition of ν in terms of the Lamé constants:

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \tag{3}$$

Plugging this result into equation (2).

$$\mu = \frac{E}{2 + \frac{\lambda}{\mu+\lambda}}$$

$$\mu = \frac{E(\mu+\lambda)}{2\mu+3\lambda}$$

Which leads to the definition of E in terms of Lamé constants:

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \tag{4}$$

2 - Plane strain and plane stress elasticity matrices in terms of λ and μ

Plane strain

Given the elasticity matrix for plane strain case:

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{5}$$

From equations (3) and (4) we can define:

$$1 - \nu = 1 - \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda + 2\mu}{2(\lambda+\mu)}$$

$$1 - 2\nu = 1 - \frac{2\lambda}{2(\lambda+\mu)} = \frac{\mu}{\lambda+\mu}$$

$$1 + \nu = 1 + \frac{\lambda}{2(\lambda+\mu)} = \frac{3\lambda + 2\mu}{2(\lambda+\mu)}$$

$$\frac{E}{(1+\nu)(1-2\nu)} = \frac{\cancel{\mu}\cancel{(2\mu+3\lambda)}}{\cancel{\mu+\lambda}} \frac{2\cancel{(\lambda+\mu)}}{\cancel{3\lambda+2\mu}} \frac{(\lambda+\mu)}{\cancel{\mu}} = 2(\lambda+\mu)$$

Plugging those results in equation (5) we have.

$$\mathbf{E} = \cancel{2(\lambda+\mu)} \begin{bmatrix} \frac{\lambda+2\mu}{\cancel{2(\lambda+\mu)}} & \frac{\lambda}{\cancel{2(\lambda+\mu)}} & 0 \\ \frac{\lambda}{\cancel{2(\lambda+\mu)}} & \frac{\lambda+2\mu}{\cancel{2(\lambda+\mu)}} & 0 \\ 0 & 0 & \frac{\mu}{\cancel{2(\lambda+\mu)}} \end{bmatrix}$$

Thus, the plane strain elasticity matrix in terms of Lamé parameters becomes:

$$\mathbf{E} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{6}$$

This result is in accordance with the constitutive equation for linear-elastic isotropic materials given by:

$$\boldsymbol{\sigma} = \lambda(tr\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon} \tag{7}$$

Plane stress

Given the elasticity matrix for plane stress case:

$$\mathbf{E} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{1}{1-\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{8}$$

From equations (3) and (4) we can define:

$$1 - \nu = 1 - \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda + 2\mu}{2(\lambda+\mu)}$$

$$1+\nu = 1 + \frac{\lambda}{2(\lambda+\mu)} = \frac{3\lambda+2\mu}{2(\lambda+\mu)}$$

$$\frac{E}{(1+\nu)} = \frac{\mu\cancel{(2\mu+3\lambda)}}{\cancel{\mu+\lambda}}\frac{\cancel{2(\lambda+\mu)}}{\cancel{3\lambda+2\mu}} = 2\mu$$

$$\frac{\nu}{1-\nu} = \frac{\lambda}{\cancel{2(\lambda+\mu)}}\frac{\cancel{2(\lambda+\mu)}}{\lambda+2\mu} = \frac{\lambda}{\lambda+2\mu}$$

Plugging those results in equation (8) we have.

$$\mathbf{E} = 2\mu \begin{bmatrix} \frac{2(\lambda+\mu)}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & 0 \\ \frac{\lambda}{\lambda+2\mu} & \frac{2(\lambda+\mu)}{\lambda+2\mu)} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Thus, the plane stress elasticity matrix in terms of Lamé parameters becomes:

$$\mathbf{E} = \frac{2\mu}{\lambda+2\mu}\begin{bmatrix} 2(\lambda+\mu) & \lambda & 0 \\ \lambda & 2(\lambda+\mu) & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{2} \end{bmatrix} \tag{9}$$

3 - Decomposition of plane strain elasticity matrix

The equation (6), for the plane strain elasticity matrix can be written as:

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

4 - Decomposition of plane strain elasticity matrix in terms of E and ν

The matrices in equation (10) can be expressed in terms of E and ν, by means of equations (1) and (2) as:

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda = \frac{E}{1+\nu}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} + \frac{\nu E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

3.2

1 - Stiffness matrices $\mathbf{K_{tri}}$ and $\mathbf{K_{bar}}$

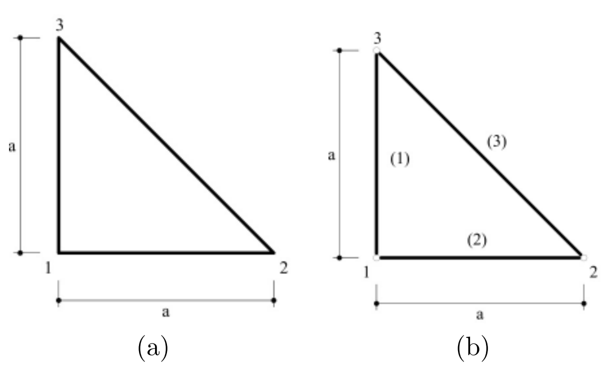

Figure 1: (a) 3-node Triangular element; (b) 3-1D bar structure

Stiffness matrix of 3-node triangular element $\mathbf{K_{tri}}$

The stiffness matrix of a triangular element $\mathbf{K_{tri}}$, Figure 1(a), is given by:

$$\mathbf{K_{tri}} = Ah\mathbf{B^T E B} \tag{12}$$

For the plane stress case with $\nu = 0$ the elasticity matrix becomes:

$$\mathbf{E} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{13}$$

Where the strain-displacement matrix $\mathbf{B}$ is given by:

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12}0 & \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{14}$$

After plugging the appropriate values for $y_{ij}4$ and x_{ij}, matrix $\mathbf{B}$ becomes:

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{15}$$

After performing the matrix multiplication of equation (12), the element stiffness matrix becomes:

$$\mathbf{K_{tri}} = \frac{Eh}{8A}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{16}$$

Given that $A = a^2/2$ and $h = a = 1$, equation (16) becomes:

$$\mathbf{K_{tri}} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ Symm. & & & & & 2 \end{bmatrix} \tag{17}$$

Stiffness matrix of 3-1D bar structure $\mathbf{K_{bar}}$

To find the 3-1D bar structure stiffness matrix $\mathbf{K_{bar}}$, of Figure 1(b), we will apply the direct stiffness method. The general form of the elemental stiffness matrix is given by:

$$\mathbf{K^{(e)}} = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ & s^2 & -sc & -s^2 \\ & & c^2 & sc \\ Symm. & & & s^2 \end{bmatrix} \tag{18}$$

Where $c = cos(\varphi)$ and $s = sin(\varphi)$, and φ is the angle between the horizontal axis of the global coordinate system and the element axial line

Element 1:

For element 1 we have:
$\varphi = \frac{\pi}{2}$, thus: $c = 0$ and $s = 1$. Further $L^{(1)} = a = 1$
From this result, equation (18), for element 1, can be written as:

$$\mathbf{K^{(1)}} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{Symm.} & & & 1 \end{bmatrix} \tag{19}$$

The expanded stiffness equations for element 1 are:

$$\begin{Bmatrix} f_{x_1}^{(1)} \\ f_{y_1}^{(1)} \\ f_{x_2}^{(1)} \\ f_{y_2}^{(1)} \\ f_{x_3}^{(1)} \\ f_{y_3}^{(1)} \end{Bmatrix} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ Symm. & & & & & A_1 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(1)} \\ u_{y_1}^{(1)} \\ u_{x_2}^{(1)} \\ u_{y_2}^{(1)} \\ u_{x_3}^{(1)} \\ u_{y_3}^{(1)} \end{Bmatrix} \tag{20}$$

Element 2:

For element 2 we have:
$\varphi = 0$, thus: $c = 1$ and $s = 0$. Further $L^{(2)} = a$
, and $A_2 = A_1$. From this result, equation (18), for element 1, can be written as:

$$\mathbf{K^{(2)}} = EA_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symm.} & & & 0 \end{bmatrix} \tag{21}$$

The expanded stiffness equations for element 2 are:

$$\begin{Bmatrix} f_{x_1}^{(2)} \\ f_{y_1}^{(2)} \\ f_{x_2}^{(2)} \\ f_{y_2}^{(2)} \\ f_{x_3}^{(2)} \\ f_{y_3}^{(2)} \end{Bmatrix} = E \begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & A_1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ Symm. & & & & & 0 \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(2)} \\ u_{y_1}^{(2)} \\ u_{x_2}^{(2)} \\ u_{y_2}^{(2)} \\ u_{x_3}^{(2)} \\ u_{y_3}^{(2)} \end{Bmatrix} \tag{22}$$

Element 3:

For element 3 we have:
$\varphi = \pi/4$, thus: $c = s = \sqrt{2}/2$. Further $L^{(3)} = a\sqrt{2} = \sqrt{2}$. From this result, equation (18), for element 3, can be written as:

$$\mathbf{K^{(3)}} = \frac{EA_3}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ \text{Symm.} & & & 1 \end{bmatrix} \tag{23}$$

The expanded stiffness equations for element 3 are:

$$
\begin{Bmatrix} f_{x_1}^{(3)} \\ f_{y_1}^{(3)} \\ f_{x_2}^{(3)} \\ f_{y_2}^{(3)} \\ f_{x_3}^{(3)} \\ f_{y_3}^{(3)} \end{Bmatrix} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ & & & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ & & & & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ Symm. & & & & & \frac{A_3}{2\sqrt{2}} \end{bmatrix} \begin{Bmatrix} u_{x_1}^{(2)} \\ u_{y_1}^{(2)} \\ u_{x_2}^{(2)} \\ u_{y_2}^{(2)} \\ u_{x_3}^{(2)} \\ u_{y_3}^{(2)} \end{Bmatrix} \tag{24}
$$

Assembling $\mathbf{K_{bar}}$

The matrices in equations (20),(22) and (24) can be summed up to form the global stiffness matrix $\mathbf{K_{bar}}$

$$
\mathbf{K_{bar}} = E \begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ & A_1 & 0 & 0 & 0 & -A_1 \\ & & A_1 + \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ & & & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ & & & & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ Symm. & & & & & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix} \tag{25}
$$

2 - Stiffness matrices equivalence

Comparing the stiffness matrices from equations (17) and (25), it can be noticed that it *is not* a set of values for A_1 and A_3 such that both are equal ($\mathbf{K_{bar}} = \mathbf{K_{tri}}$). However,if one wants to find values for $A1$ and $A3$ such that the matrices are more similar, a option would be choosing $A1 = 3/4$ and $A_3 = A_1\sqrt{2}/2$ or $A_3 = 0$ (which is not a physical valid value). Taking the first option for A_3 the $\mathbf{K_{bar}}$ matrix becomes.

$$
\mathbf{K_{bar}} = \frac{E}{4} \begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & -3 \\ & & 4 & 1 & -1 & -1 \\ & & & 1 & -1 & -1 \\ & & & & 1 & 1 \\ Symm. & & & & & 4 \end{bmatrix} \tag{26}
$$

3 - Physical point of view of previous results and $\nu \neq 0$ case

The fact that the two matrices can not be made equal is actually a expected value, as each matrix comes from a different model. Thinking from the energy point of view, the matrix $\mathbf{K_{tri}}$ has more no-zero elements which is a result of considering more pairs of forces (f_i) and DOF displacement (u_j) to make up the balance between external work and internal energy. For example, the first row of the matrix $\mathbf{K_{tri}}$, all elements, except for the one in last column (K_{16}), are no-zero valued, which means that the work done by a force (f_{x1}) on the first DOF of the system (Which corresponds to the displacement u_{x1}) will be stored as internal energy by the displacements in all DOFs (except last DOF related with (u_{y3}). For the 3-1D bar structure case, the model is such that those structural bars are only able to withstand axial load. This way, as an example, the first row of the $\mathbf{K_{bar}}$ has only two no-zero entrances, which are related with the energy stored by the two axial displacements of the element 2. It means that the work done by a force (f_{x1}) on the first DOF of the system (Which corresponds to the displacement u_{x1}) will be stored as internal energy by the axial displacements of element 2 (u_{x1} and u_{x2}), as for this bar element, load in x direction can not produce displacements in y direction, thus no energy change takes place in y direction displacement in this load case. Roughly speaking, it can be stated that the 3-node Element structure is stiffer than the 3-1D bar one, because for any applied load (Or energy add by work), there are more degree of freedoms receiving the energy or resisting this external load.

Given those facts, in general the $\mathbf{K_{bar}}$ will be more sparse than $\mathbf{K_{tri}}$ due to it restriction on the way forces and DOF displacements are related.
Further, if the assumption of $\nu = 0$ was not made for the 3-node triangular element case, more terms with non-zero values would appear in the matrix due to the Poisson effect, as it can be seen below.

$$\mathbf{K_{tri}} = \frac{Eh}{8A(1-\nu^2)} \begin{bmatrix} 3-\nu & 1-3\nu & -2 & \nu-1 & \nu-1 & -2\nu \\ & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ & & 2 & 0 & 0 & 2\nu \\ & & & 1-\nu & 1-\nu & 0 \\ & & & & 1-\nu & 0 \\ Symm. & & & & & 2 \end{bmatrix} \tag{27}$$

As an example, the third line of the $\mathbf{K_{tri}}$ matrix, equation (17) for the case with $\nu = 0$, is relating the force f_{x2} only with the displacements in x direction of nodes 1 and 2 (u_{x1} and u_{x2}), thus the energy by the external work is being stored only by this 2 internal work components. However, if Poisson effects were considered ($\nu \neq 0$) the work done by f_{x2} would have some part stored as internal energy by the vertical displacements u_{y1} and u_{y3}, as it can be noticed in equation (27) where the components K_{32} and K_{36} are non-zero. This logic can be extended for all force components and degree of freedoms of the structure.