Computational Structural Mechanics and Dynamics

Assignment 3, Trond Jørgen Opheim

3.1 $\lambda = \frac{Ev}{(1+v)(1-2v)}$ $\mu = \frac{E}{2(1+v)}$

1. E and v in terms of λ and μ

$\lambda(1-2v) = 2 * \frac{E}{2(1+v)} v$

$\lambda(1-2v) = 2\mu v$ $\boldsymbol{v = \frac{\lambda}{2(\lambda+\mu)}}$

$E = \mu * 2(1+v)$

$E = 2\mu\left(\frac{2(\lambda+\mu)}{2(\lambda+\mu)} + \frac{\lambda}{2(\lambda+\mu)}\right)$

$E = \frac{2\mu\lambda+2\mu^2+\mu\lambda}{\lambda}$ $\boldsymbol{E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}$

2. Elastic matrixes in terms of λ and μ

See attached picture for calculations. The results are the following:

Plane stress: $\boldsymbol{E} = \frac{\mu(3\lambda+2\mu)}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix} 4(\lambda+\mu) & 2\lambda & 0 \\ 2\lambda & 4(\lambda+\mu) & 0 \\ 0 & 0 & \lambda+2\mu \end{bmatrix}$

Plane strain: $\boldsymbol{E} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$

3. Lamé-splitting of the plain strain elastic matrix

$$\boldsymbol{E} = \boldsymbol{E_\lambda} + \boldsymbol{E_\mu} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

4. $\boldsymbol{E_\lambda}$ and $\boldsymbol{E_\mu}$ in terms of E and v.

$$\boldsymbol{E_\lambda} + \boldsymbol{E_\mu} = \frac{Ev}{(1+v)(1-2v)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{E}{2(1+v)}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② Elastic matrix in terms of λ and μ

Plane stress

$$\mathbb{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}{1-\frac{\lambda^2}{2^2(\lambda+\mu)^2}}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{2} \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\cdot\frac{2^2(\lambda+\mu)^2}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix} & & \\ & & \end{bmatrix}$$

$$= \frac{4\mu(3\lambda+2\mu)(\lambda+\mu)}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{4(\lambda+\mu)} \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix} 4(\lambda+\mu) & 2\lambda & 0 \\ 2\lambda & 4(\lambda+\mu) & 0 \\ 0 & 0 & \lambda+2\mu \end{bmatrix}$$

Plane strain

$$\mathbb{E} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$= \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}{1-\nu-2\nu^2}\begin{bmatrix} & & \\ & & \end{bmatrix}$$

$$= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\cdot\frac{2(\lambda+\mu)^2}{3\lambda\mu+2\mu^2}\begin{bmatrix} & & \\ & & \end{bmatrix}$$

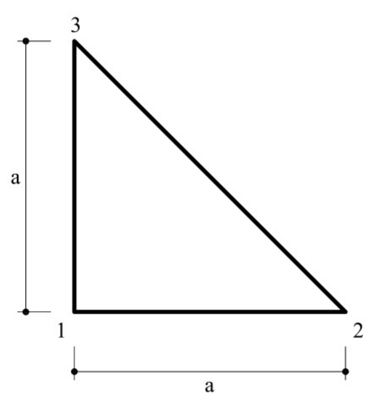
$$= 2(\lambda+\mu)\begin{bmatrix} \frac{2(\lambda+\mu)-\lambda}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & \\ & & \\ & & \frac{1}{2}\left(\frac{2(\lambda+\mu)}{2(\lambda+\mu)} - 2\cdot\frac{\lambda}{2(\lambda+\mu)}\right) \end{bmatrix}$$

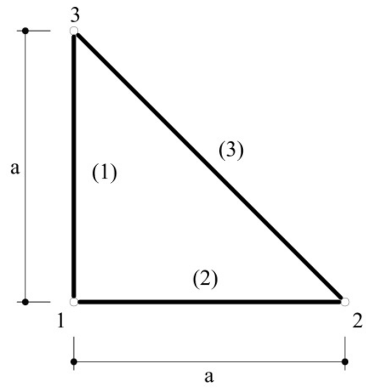
$$= \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

③

$$\mathbb{E} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

3.2





1. Stiffness matrixes $\boldsymbol{K}_{tri}$ and $\boldsymbol{K}_{bar}$ for the two models.

$\boldsymbol{K}_{tri}$: from the slides “CSMD_5-Linear_triangle” on CIMNE I use the property

$$\mathbf{K}^e = \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^e} h \, d\Omega$$

$$= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h \, d\Omega$$

and write a simple script in Matlab to compute the stiffness matrix for the linear Turner triangle. The script is attached at the next page and the resulting stiffness matrix is

$$\boldsymbol{K}_{tri} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\boldsymbol{K}_{bar}$ **:** to compute the stiffness matrix for the system of bars I also write a simple script in Matlab that computes the stiffness matric for each element, and then I assemble them after their numbering of nodes and bars. This is also attached at the next page.

$$\boldsymbol{K}_{bar} = \frac{E}{4} \begin{bmatrix} 4A_2 & 0 & -4A_2 & 0 & 0 & 0 \\ 0 & 4A_1 & 0 & 0 & 0 & -4A_1 \\ -4A_2 & 0 & \sqrt{2}A_3 + A_2 & -\sqrt{2}A_3 & -\sqrt{2}A_3 & \sqrt{2}A_3 \\ 0 & 0 & -\sqrt{2}A_3 & \sqrt{2}A_3 & \sqrt{2}A_3 & -\sqrt{2}A_3 \\ 0 & 0 & -\sqrt{2}A_3 & \sqrt{2}A_3 & \sqrt{2}A_3 & -\sqrt{2}A_3 \\ 0 & -4A_1 & \sqrt{2}A_3 & -\sqrt{2}A_3 & -\sqrt{2}A_3 & \sqrt{2}A_3 + A_1 \end{bmatrix}$$

Script for computing K_tri:

```
clear
a=1;
h=1;
A=0.5*a^2;

syms E
x1=0;   %x-position node 1
y1=0;   %y-position node 1
x2=a;   %x-posotion node 2
y2=0;   %y-position node 2
x3=0;   %x-position node 3
y3=a;   %y-position node 3

x13=x1-x3;
x21=x2-x1;
x32=x3-x2;

y23=y2-y3;
y31=y3-y1;
y12=y1-y2;

E_mat=[E 0 0;0 E 0;0 0 0.5*E];  %v=0
B=[y23 0 y31 0 y12 0;0 x32 0 x13 0 x21;x32 y23 x13 y31 x21 y12];
B_=transpose(B);

K_tri=(h/(4*A))*B_*E_mat*B
```

Script for computing K_bar:

```
clear

syms E A
angle=3*pi/4; %angle each bar. In this case bar 3
c=cos(angle);
s=sin(angle);

K=E*A*[c^2 s*c -c^2 -c*s;s*c s^2 -s*c -s^2;-c^2 -s*c c^2 s*c;-s*c -s^2 s*c s^2];
```

2. $\boldsymbol{K_{tri}} = \boldsymbol{K_{bar}}$?

These two stiffness matrices can obviously not be the same because they represent two different systems. But they can be made to have more similar values by changing the actual parameters. One way to do this is to make equations with values from one matrix and from the other matrix with the corresponding entry. For example:

Entry 1,1: $3 = 4A_2$

Entry 1,3: $-2 = -4A_2$

Entry 3,3: $2 = \sqrt{2}A_3 + A_2$

This will make a set of equations, which can not be solved because these are two different systems, BUT they can be used to make the stiffness matrices more similar.

3. Why these stiffness matrices are not equal

As said in the previous section, the two matrices represent the stiffness of two different systems. For the case with the solid linear triangle we have stiffness contribution from the whole continuous area. In the case of the bar structure, we only have contribution to stiffness along the bars. By physical sense this could be explained by that the solid triangle will have contribution in the whole area, but as for the bars they could only take up axial forces in compression and tension.

4. v≠0

When v≠0 we will have nonzero values in entries (1,2), (2,1). This will affect the stiffness matrices which is a product of a matrix with geometric properties and the elastic matrix. With this affection we would, normally, have more nonzero entries in the stiffness matrix than if v=0.