COMPUTATIONAL SUCTURAL MECHANICS AND DYNAMICS

Master of Science in Computational Mechanics/Numerical Methods

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Assignment 3:

1. Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress and plane strain as given in any textbook on elasticity are:

$$plane\ stress: \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix}$$

$$plane\ strain: \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix}$$

a) Show that the constitutive matrix of plane strain can be formally obtained by replacing E by a fictitious modulus E^* and ν by a fictitious Poisson's ratio ν^* in the plane stress constitutive matrix. Find the expression of E^* and ν^* in terms of E and ν.

Three relations must be satisfied:

$$\frac{E^*}{1-(\nu^*)^2} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*\nu^*}{1-(\nu^*)^2} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*(1-\nu^*)}{2(1-(\nu^*)^2)} = \frac{E}{2(1+\nu)}$$

The first relation leads to:

$$E^* = E\frac{(1-(\nu^*)^2)(1-\nu)}{(1+\nu)(1-2\nu)}$$

Substituting the first into the second relation:

$$\frac{E(1-(\nu^*)^2)\nu^*}{1-(\nu^*)^2} = E\nu\frac{(1+\nu)(1-2\nu)}{(1+\nu)(1-2\nu)(1-\nu)} \rightarrow \nu^* = \frac{\nu}{1-\nu}$$

Back to the first relation, substituting the result obtained:

$$E^* = E\frac{\left(1-\left(\frac{\nu}{1-\nu}\right)^2\right)(1-\nu)}{(1+\nu)(1-2\nu)} = E\frac{(1-\nu)^2-\nu^2}{(1-\nu)^2}\frac{(1-\nu)}{(1+\nu)(1-2\nu)} =$$

$$= E\frac{1-2\nu}{(1-\nu)(1+\nu)(1-2\nu)} = \frac{E}{1-\nu^2}$$

The last step is to check that the last relation holds using the two previous relation:

$$\frac{E^*(1-\nu^*)}{2(1-(\nu^*)^2)} = \frac{E}{1-\nu^2}\left(1-\frac{\nu}{1-\nu}\right)\frac{1}{2\left(1-\left(\frac{\nu}{1-\nu}\right)^2\right)}$$

$$= \frac{E}{1-\nu^2}\frac{(1-2\nu)}{1-\nu}\frac{(1-\nu)^2}{2(1-2\nu)} = \frac{E}{2}\frac{1-\nu}{1-\nu^2} = \frac{E}{2(1+\nu)}$$

Which is exactly the third relation. So the constitutive matrix of plane strain can be obtained from the plane stress matrix substituting:

$$E^* = \frac{E}{1-\nu^2}, \nu^* = \frac{\nu}{1-\nu}$$

b) Do also the inverse process: go from plane strain to plane strain by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix.

To obtain the inverse relation, the only thing to do is to invert the two relations for E and ν:

$$E = \frac{\hat{E}}{1-\hat{\nu}^2}, \nu = \frac{\hat{\nu}}{1-\hat{\nu}}$$

The inverse of the second relation is computed first:

$$\nu = \frac{\hat{\nu}}{1-\hat{\nu}} \rightarrow \nu(1-\hat{\nu}) - \hat{\nu} = 0 \rightarrow \hat{\nu} + \nu\hat{\nu} - \nu = \hat{\nu}(1+\nu) - \nu = 0$$

$$\hat{\nu} = \frac{\nu}{1+\nu}$$

Substituting in the first equation:

$$E = \frac{\hat{E}}{1-\hat{\nu}^2} = \hat{E}\frac{1}{1-\left(\frac{\nu}{1+\nu}\right)^2} = \hat{E}\frac{(1+\nu)^2}{1+2\nu} \rightarrow \hat{E} = E\frac{1+2\nu}{(1+\nu)^2}$$

The relations to transform the plane strain constitutive matrix to plane stress are:

$$\hat{E} = E\frac{1+2\nu}{(1+\nu)^2}, \hat{\nu} = \frac{\nu}{1+\nu}$$

2. **In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimensions of stress and are related to E and ν by:**

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = G = \frac{E}{2(1+\nu)}$$

   a) **Find the inverse relations for E, ν in terms of λ, μ:**

   From the second relation:

$$E = 2\mu(1+\nu)$$

   Substituting on the first relation:

$$\lambda = \frac{2\mu(1+\nu)\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{1-2\nu} \to \lambda - 2\lambda\nu - 2\mu\nu = 0 \to \lambda - 2\nu(\lambda+\mu) = 0$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

   Back to the previous expression:

$$E = 2\mu\left(1 + \frac{\lambda}{2(\lambda+\mu)}\right) = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$$

   The inverse equations are:

$$E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}, \nu = \frac{\lambda}{2(\lambda+\mu)}$$

   b) **Express the elastic matrix for plane stress and plane strain cases in term of λ, μ:**

   The plane stress constitutive matrix is:

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

   Substituting:

$$\mu\frac{3\lambda+2\mu}{\lambda+\mu}\frac{1}{1-\left(\frac{\lambda}{2(\lambda+\mu)}\right)^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{\lambda}{2(\lambda+\mu)}}{2} \end{bmatrix} =$$

$$\mu\frac{3\lambda+2\mu}{\lambda+\mu}\frac{4(\lambda+\mu)^2}{4(\lambda+\mu)^2-\lambda^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\lambda+\mu)} & 0 \\ \frac{\lambda}{2(\lambda+\mu)} & 1 & 0 \\ 0 & 0 & \frac{2(\lambda+\mu)-\lambda}{4(\lambda+\mu)} \end{bmatrix} =$$

$$4\mu\frac{3\lambda+2\mu}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix}\lambda+\mu & \frac{\lambda}{2} & 0\\ \frac{\lambda}{2} & \lambda+\mu & 0\\ 0 & 0 & \frac{\lambda+2\mu}{4}\end{bmatrix}$$

c) Split the stress-strain matrix $\boldsymbol{E}$ of plane strain as:

$$\boldsymbol{E}=\boldsymbol{E}_{\lambda}+\boldsymbol{E}_{\mu}$$

In which $\boldsymbol{E}_{\lambda}$ and $\boldsymbol{E}_{\mu}$ contain only $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. This is the Lamé $\{\boldsymbol{\lambda},\boldsymbol{\mu}\}$ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements:

The plane strain constitutive matrix is:

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

Substituting:

$$\mu\frac{3\lambda+2\mu}{\lambda+\mu}\frac{1}{1+\frac{\lambda}{2(\lambda+\mu)}}\frac{1}{1-\frac{\lambda}{(\lambda+\mu)}}\begin{bmatrix}1-\frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & 1-\frac{\lambda}{2(\lambda+\mu)} & 0\\ 0 & 0 & \frac{1-\frac{\lambda}{(\lambda+\mu)}}{2}\end{bmatrix}=$$

$$\mu\frac{3\lambda+2\mu}{\lambda+\mu}\frac{2(\lambda+\mu)}{2(\lambda+\mu)+\lambda}\frac{(\lambda+\mu)}{(\lambda+\mu)-\lambda}\begin{bmatrix}\frac{\lambda+2\mu}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0\\ \frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda+2\mu}{2(\lambda+\mu)} & 0\\ 0 & 0 & \frac{\mu}{2(\lambda+\mu)}\end{bmatrix}=$$

$$\begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}=\begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

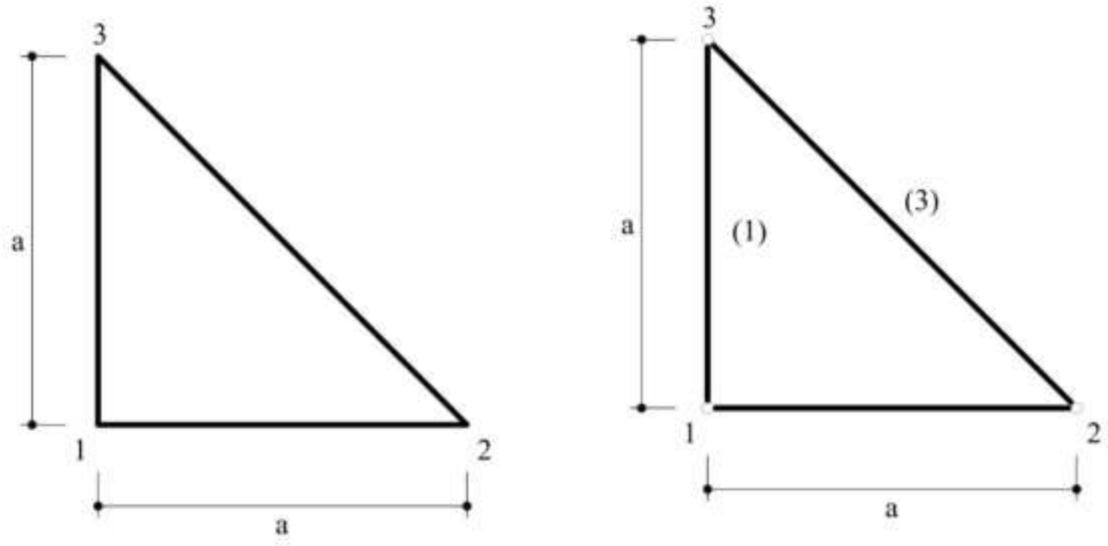
d) Express $\boldsymbol{E}_{\lambda}$ and $\boldsymbol{E}_{\mu}$ also in terms of $\boldsymbol{E}$ and $\boldsymbol{\nu}$:

The only step is to substitute the Lamé parameters using the given relations:

$$\begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}=$$

$$\frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}+\frac{E}{2(1+\nu)}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

3. **Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:**
    - **A plane linear Turner triangle with the same dimensions.**
    - **A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.**



a) **Calculate the stiffness matrix K^e for both models.**

The calculation of the Turner model is done as follows:

$$\boldsymbol{K^e} = \frac{h}{4A}\boldsymbol{B^T E B}$$

$$= \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Substituting the numerical values:

$$\boldsymbol{K^e} = \frac{E}{2(1-\nu^2)}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} =$$

$$K^e = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

In the case of $\nu = 0$ the stiffness matrix is the following:

$$K^e = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

For the calculation of the bar triangle the three elemental matrices are first calculated:

$$K^1 = EA\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^2 = EA\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = \frac{EA'}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Taking $A^* = \frac{A'}{2\sqrt{2}}$, the assembly process is done:

$$K = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + A^* & A^* & -A^* & -A^* \\ 0 & 0 & A^* & A^* & -A^* & -A^* \\ 0 & 0 & -A^* & -A^* & A^* & A^* \\ 0 & -A & -A^* & -A^* & A^* & A + A^* \end{bmatrix}$$

b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

The two matrices have not similar structure at all. Even for the only diagonal terms is hard to find a value of A and A^* that make both matrices similar.

Taking the K_{11} and K_{22} terms we could arrive to the conclusion that $A = \frac{3}{4}$. But then, looking at K_{33} and K_{66} terms this would mean that the value of A^* should be negative making no physical sense.

c) Why these two stiffness matrices are not equivalent? Fins a physical explanation.

The two matrices present their major differences on the off-diagonal terms. That is due to the fact that in the bar case, there is no distortion energy stored in the process of deformation. Only the axial tension and compression produces reaction terms while in the Turner element distortion plays an important role.

d) Solve question a) considering $\boldsymbol{\nu \neq 0}$ and extract some conclusions.

This case was calculated already in question a):

$$\boldsymbol{K^e} = \frac{E}{2(1-\nu^2)} \begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

When taking into account the effect of Poisson's ratio it is seen that the stiffness on the diagonal terms is reduced. This is because the deformation due to the Poisson effect is in the same direction than the imposed via the external forces. To maintain the equilibrium, other terms (most of off-diagonal terms) increase with the Poisson's ratio.