Universitat Politècnica de Catalunya - Master of science in computational mechanics

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

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Assignment 3

1a

plane stress $\Longleftrightarrow$ plane strain

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E^*}{1-\nu^{*2}}\begin{bmatrix}1 & \nu^* & 0\\ \nu^* & 1 & 0\\ 0 & 0 & \frac{1-\nu^*}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix}$$

if by using fictitious E^* and ν^* on the plane stress relation we get plane strain, we'd have the following system

$$\begin{cases}\dfrac{E^*}{(1+\nu^*)(1-\nu^*)} = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} & (1)\\[2ex] \dfrac{E^*\nu^*}{(1+\nu^*)(1-\nu^*)} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} & (2)\\[2ex] \dfrac{E^*}{2(1+\nu^*)} = \dfrac{E}{2(1+\nu)} & (3)\end{cases}$$

substituting (3) on (2)

$$\frac{\cancel{E}\nu^*}{\cancel{(1+\nu)}(1-\nu^*)} = \frac{\cancel{E}\nu}{\cancel{(1+\nu)}(1-2\nu)}$$

$$\nu^*(1-2\nu) = \nu(1-\nu^*) \Rightarrow \boxed{\nu^* = \frac{\nu}{1-\nu}}$$

substituting the relation found for ν^* on (1)

$$\frac{E^*}{\left(1+\frac{\nu}{1-\nu}\right)\left(1-\frac{\nu}{1-\nu}\right)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \Rightarrow$$

$$\Rightarrow \frac{E^*}{\left[\frac{1-2\nu}{(1-\nu)^2}\right]} = \frac{E\cancel{(1-\nu)}}{(1+\nu)\cancel{(1-2\nu)}} \Rightarrow \boxed{E^* = \frac{E}{1-\nu^2}}$$

1b to do the inverse process we get the same system, but the fictitious variables are inserted on the plane strain relations.

substituting (3) on (2) on the modified system yields:

$$\frac{E\nu^*}{(1+\nu)(1-2\nu^*)} = \frac{E\nu}{(1+\nu)(1-\nu)}$$

$$\nu^*(1-\nu) = \nu(1-2\nu^*)$$

$$\nu^*(1+\nu) = \nu$$

$$\boxed{\nu^* = \frac{\nu}{1+\nu}}$$

substituting the relation found for ν^* on the modified (1)

$$\frac{E^*\left(1-\frac{\nu}{1+\nu}\right)}{\left(1+\frac{\nu}{(1+\nu)}\right)\left(1-\frac{2\nu}{1+\nu}\right)} = \frac{E}{(1+\nu)(1-\nu)}$$

$$\frac{E^*\left(\frac{1}{1+\nu}\right)}{\frac{(1+2\nu)(1-\nu)}{(1+\nu)^2}} = \frac{E}{(1+\nu)(1-\nu)} \Rightarrow \boxed{E^* = \frac{E(1+2\nu)}{(1+\nu)^2}}$$

2a Lamé constants $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ (1) ; $\mu = \frac{E}{2(1+\nu)}$ (2)

substituting (2) in (1) : $\lambda = \frac{2\mu\nu}{1-2\nu} \Rightarrow \lambda - 2\lambda\nu = 2\mu\nu \Rightarrow \boxed{\nu = \frac{\lambda}{2(\lambda+\mu)}}$ (3)

substituting (3) in (2) : $\mu = \frac{E}{2\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)} = \frac{E}{2+\frac{\lambda}{\lambda+\mu}} = \frac{E(\lambda+\mu)}{2\mu+3\lambda} \Rightarrow$

$$\Rightarrow \boxed{E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}$$

2b The elastic matrix for plane stress in terms of λ and μ :

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \Rightarrow \frac{E}{1-\nu^2} = \frac{2\mu}{1-\frac{\lambda}{2(\lambda+\mu)}} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \; ; \; \frac{E\nu}{1-\nu^2} = \frac{2\mu\lambda}{\lambda+2\mu} \; ; \; \frac{E(1-\nu)}{2(1-\nu^2)} = \mu$$

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{2\mu}{\lambda+2\mu}\begin{bmatrix} 2(\lambda+\mu) & \lambda & 0 \\ \lambda & 2(\lambda+\mu) & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{2} \end{bmatrix}$$

The elastic matrix for plane strain in terms of λ and μ:

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \Rightarrow \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{2\mu}{1-2\nu} - \lambda = \frac{2\mu}{1-\frac{\lambda}{\lambda+\mu}} - \lambda = 2(\lambda+\mu) - \lambda = \lambda + 2\mu$$

$$\frac{E\nu}{(1+\nu)(1-2\nu)} = \lambda \; ; \; \frac{E(1-2\nu)}{(1+\nu)(1-2\nu)2} = \mu$$

$$\Rightarrow \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

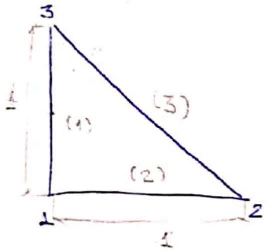
2c split the plane strain matrix in $E = E_\mu + E_\lambda$

$$\begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_\lambda} + \underbrace{\begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}_{E_\mu}$$

[redacted].2d

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad E_\mu = \frac{E}{1+\nu}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

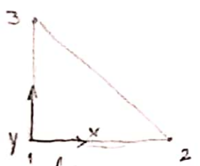
3a



for a set of 3 bar elements :

$$K^{(1)} = EA\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} ; \quad K^{(2)} = EA\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \quad K^{(3)} = \frac{EA'}{\sqrt{2}}\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

and the global stiffness :



$$K = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+A'/2\sqrt{2} & -A'/2\sqrt{2} & -A'/2\sqrt{2} & A'/2\sqrt{2} \\ 0 & 0 & -A'/2\sqrt{2} & A'/2\sqrt{2} & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ 0 & 0 & -A'/2\sqrt{2} & A'/2\sqrt{2} & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ 0 & -A & A'/2\sqrt{2} & -A'/2\sqrt{2} & -A'/2\sqrt{2} & A+A'/2\sqrt{2} \end{bmatrix}$$ //

for a plane linear triangle under plane stress

$$K = \frac{1}{4A_t}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} ; \quad \text{with } A_t = \frac{1}{2}$$

$$K = \frac{E}{4(1-\nu^2)}\begin{bmatrix} 3-\nu & \nu+1 & -2 & \nu-1 & \nu-1 & -2\nu \\ \nu+1 & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ -2 & -2\nu & 2 & 0 & 0 & 2\nu \\ \nu-1 & \nu-1 & 0 & 1-\nu & 1-\nu & 0 \\ \nu-1 & \nu-1 & 0 & 1-\nu & 1-\nu & 0 \\ -2\nu & -2 & 2\nu & 0 & 0 & 2 \end{bmatrix}$$ //

3b

with $\nu=0$, we can find values for A and A' that approximate both stiffness matrices. Elements of the principal diagonal K_{11}, K_{22}, K_{44} and K_{55} can be the same for both methods of

$$A = \frac{3}{4} \text{ and } A' = \frac{\sqrt{2}}{2} ;$$ yielding:

$$K_{bar} = \frac{E}{4}\begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & -3 \\ & & 4 & -1 & -1 & 1 \\ & & & 1 & 1 & -1 \\ & & & & 1 & -1 \\ \text{sym.} & & & & & 4 \end{bmatrix} \qquad K_{tri} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ \text{sym.} & & & & & 2 \end{bmatrix}$$

$A = \frac{3}{4}$; $A' = \frac{\sqrt{2}}{2}$

similarly elements K_{33}, K_{44}, K_{55} and K_{66} can be equaled if $A = \frac{1}{4}$ and $A' = \frac{\sqrt{2}}{2}$. However, none of these cases makes the matrices to be completely equal.

3c The stiffness matrices are not the same because of the assumptions made for each model. The bar elements are 1D objects and can, therefore, only offer resistance in one direction. Each equation of the linear system $\underline{\underline{K}}\,\underline{u} = \underline{f}$ will only contain displacements of nodes connected to the evaluated node on the evaluated direction ($\frac{3E}{4}(u_{1x} - u_{2x}) = f_{1x}$, for example). On the other hand, the turner triangle is a 2D element and has, thus, a connectivity among all the points of the object. The resistance against a horizontal force on a node will no longer come only from the bars horizontally connected to it, but from the whole object as the stiffness matrix itself shows.

3d As already derived in exercise 3a, the stiffness matrix with $\nu \neq 0$ for the Turner triangle goes even further away from the bar stiffness. Adding the Poisson effect, the element has further displacement responses in the perpendicular direction of the stresses, making it even more difficult for a 1D model to capture this behaviour.