Assignment 3

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Assignment 3.1

1) To calculate $\boldsymbol{K^e}$, the following formula is used:

$$K^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Note that the notation for x and y are made such that $x_{ij} = x_i - x_j$. The resultant of this matrix is a 6×6 matrix. Having all the given values, the calculation goes as follows:

$$A = \frac{1}{2} det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = 2$$

$$K^e = \frac{1}{8}\begin{bmatrix} -100 & -25 & -50 \\ -25 & -100 & -50 \\ 200 & 50 & -100 \\ -50 & -200 & 100 \\ -100 & -25 & 150 \\ 75 & 300 & -50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K^e = \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & 25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ 25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix}$$

$$K^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & 3.125 \\ & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ & & 75 & -37.5 & -62.5 & 31.25 \\ & & & 75 & 43.75 & -87.5 \\ & symm & & & 68.75 & -28.125 \\ & & & & & 118.75 \end{bmatrix}$$

2) summing rows 1, 3 and 5 (for the x coordinates)defined as X, then rows 2, 4 and 6 (for the y coordinates)defined as Y:

$$X = 18.75 + 9.375 - 12.5 - 6.25 - 6.25 + 3.125 - 12.5 + 6.25 + 75 - 37.5 - 62.5$$
$$+31.25 - 6.25 - 15.625 - 62.5 + 43.75 + 68.75 - 28.125 = 0$$
$$Y = 9.375 + 18.75 + 6.25 + 12.5 - 15.625 - 31.25 - 6.25 + 12.5 - 37.5 + 75 + 43.75$$
$$-87.5 + 3.125 - 31.25 + 31.25 - 87.5 - 28.125 + 118.75 = 0$$

It is also noticeable that the sum of each row alone is also zero. This is because there are no boundary conditions on this plane stress triangle and no forces, therefore the body is in equilibrium.

Assignment 3.2

a)

The following part of the problem will be numbered as follows:
1) one triangular element with same dimensions of the given problem
2) three bar element with areas $A_1 = A_2$ and A_3 for elements 1,2 and 3 respectively.

1) the matrix $\boldsymbol{K}$ will be calculated the same way it was calculated in the previous exercise.
For $v = 0$, Young's modulus will be as follows:

$$\boldsymbol{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix}$$

Therefore, knowing that $a = h = 1$, the stiffness matrix will be as follows:

$$\boldsymbol{K} = \frac{1}{2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{1}{2} \begin{bmatrix} -E & 0 & -\frac{E}{2} \\ 0 & -E & -\frac{E}{2} \\ E & 0 & 0 \\ 0 & 0 & \frac{E}{2} \\ 0 & 0 & \frac{E}{2} \\ 0 & E & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{E}{2} \begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ & 1.5 & 0 & -0.5 & -0.5 & -1 \\ & & 1 & 0 & 0 & 0 \\ & & & 0.5 & 0.5 & 0 \\ & & & & 0.5 & 0 \\ & & & & & 1 \end{bmatrix}$$

2) To calculate the stiffness matrix, the direct stiffness method is used.

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

the global x-y coordinates will be taken such that the x-axis is horizontal and the y-axis is vertical.

Note that $A_1 = A_2 = A$. The elements are chosen as follows:
(1): 3→1
(2): 1→2
(3): 2→3

$$K^1 = EA \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^2 = EA \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = \frac{EA_3}{\sqrt{2}} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Now the construction of the global stiffness matrix will be followed:

$$K = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & A \\ -A & 0 & A + 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} \\ 0 & 0 & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} \\ 0 & 0 & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} \\ 0 & A & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -A + 0.5A_3/\sqrt{2} \end{bmatrix}$$

b)

Both matrices are different and cannot be written as one function of the other as there are non zero terms in matrix that are in the position of zero terms in the other (like in the 2nd and 3rd rows)

c)

These two matrices are not equivalent because they were solved in two different concepts. The first one is that it is represented as a constant thickness, one triangular element. The other is that each side of the triangle is replaced by a bar element with different cross sectional areas, and therefore leaving a void between the bars.

d)

1) For $\upsilon \neq 0$, Young's modulus will be as follows:

$$\boldsymbol{E} = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}$$

Therefore, knowing that $a = h = 1$, the stiffness matrix will be as follows:

$$\boldsymbol{K} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{E}{2(1-\upsilon^2)}\begin{bmatrix} -1 & -\upsilon & \frac{\upsilon-1}{2} \\ -\upsilon & -1 & \frac{\upsilon-1}{2} \\ 1 & \upsilon & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \\ 0 & 0 & \frac{1-\upsilon}{2} \\ \upsilon & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{E}{2(1-\upsilon^2)}\begin{bmatrix} \frac{3-\upsilon}{2} & \frac{1+\upsilon}{2} & -1 & \frac{\upsilon-1}{2} & \frac{\upsilon-1}{2} & -\upsilon \\ & \frac{3-\upsilon}{2} & -\upsilon & \frac{\upsilon-1}{2} & \frac{\upsilon-1}{2} & -1 \\ & & 1 & 0 & 0 & \upsilon \\ & & & \frac{1-\upsilon}{2} & \frac{1-\upsilon}{2} & 0 \\ & & & & \frac{1-\upsilon}{2} & 0 \\ & & & & & 1 \end{bmatrix}$$

2)

$$K = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & A \\ -A & 0 & A+0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} \\ 0 & 0 & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} \\ 0 & 0 & -0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} \\ 0 & A & 0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -0.5A_3/\sqrt{2} & -A+0.5A_3/\sqrt{2} \end{bmatrix}$$

Note that υ is not involved in the calculations in the direct stiffness method, therefore the answer is intact.

The two stiffness matrices are even more different now than before due to the integration of the poison's ratio. Therefore one should choose the method depending on the given of the problem. In this case, the first method is the one to use.