Computational Structural Mechanics and Dynamics
Assignment 3

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Assignment 3.1

Compute the entries of Ke for the following plane stress triangle:

$$x1 = 0, y1 = 0, x2 = 3, y2 = 1, x3 = 2, y3 = 2$$
$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, h = 1 \tag{1}$$

Answer

In order to describe the triangular coordinates in the Cartesian ones, it is needed to make a transformation. This transformation is done by:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{23} \\ 2A_{31} & y_{31} & x_{31} \\ 2A_{12} & y_{12} & x_{12} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{2}$$

In which $x_{jk} = x_j - x_k$ and $y_{jk} = y_j - y_k$ and the A_{jk} is the area subtended by corners j,k and the origin of the system. So now the displacements field can be computed as:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{3}$$

The shape functions in the linear triangle elements are the triangular coordinates, so the B matrix becomes:

$$B = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{4}$$

The problem to be solved is $Ku = f$ and the matrix K is defined as:

$$K^e = \int_{\Omega^e} h B^T E B d\Omega \tag{5}$$

Considering that E and B are constant in the element, they can be extracted from the integral. For this problem, the matrix B will be:

$$B = \frac{1}{4} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix} \tag{6}$$

Considering that h is constant, $\int_{\Omega^e} h d\Omega = A$. The stiffness matrix is:

$$K^e = \frac{1}{8}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix} \tag{7}$$

Computing the matrix products:

$$K = \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix} \tag{8}$$

The results are verified as $K_{11} = 18,75$ and $K_{66} = 118,75$.

Show that the sum of the rows (and columns) 1, 3 and 5 of Ke as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

Answer

To compute this matrix, a function has been assigned in order to describe the connection between each node: the shape functions. These functions have the values 0 in all the nodes except of the referred one in which is 1. It means that summing the values on these rows (or columns), the results has to be zero as it is summing his relations with the other nodes, described by the shape functions.

Assignment 3.2

Answer

a)

The stiffness matrix of the first model, the plane linear Turner triangle, can be computed as in the first part of the assignment. So the resultant B matrix is:

$$B = \frac{1}{2\frac{a^2}{2}}\begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix} \tag{9}$$

The matrix E is:

$$E = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{10}$$

and if $\nu = 0$, $a = h = 1$ the matrix K will take the form of:

$$K = \frac{E}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{11}$$

$$K = \frac{E}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

$$K = E \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{13}$$

To compute the stiffness matrix of the second model, the previous method can't be used, but it is needed to work bar by bar. That's because the section of each bar is different while in the first model the section was the same everywhere in the triangle. Moreover the two structures are different, one is a unique structure, the second one are three bars linked by three hinges.

For the single elements we can use, from the previous classes, the To represent the local coordinates of each element in the global ones, it is needed to introduce a relation:

$$\begin{aligned} u_x^e &= u_x^{glob} c + u_y^{glob} s \\ u_y^e &= -u_x^{glob} s + u_y^{glob} c \end{aligned} \tag{14}$$

in which $c = \cos\alpha$ and $s = \sin\alpha$. Writing it in matrix formulation:

$$\begin{bmatrix} u_{xi}^e \\ u_{yi}^e \\ u_{xj}^e \\ u_{yj}^e \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi}^{glob} \\ u_{yi}^{glob} \\ u_{xj}^{glob} \\ u_{yj}^{glob} \end{bmatrix}$$

The matrix multiplying represented above is the T matrix that transform the local in global coordinates. Writing also the stiffness matrix in the global system for each element:

$$K^{glob} = (T)^T K^e T \tag{15}$$

This will lead to a rotated stiffness matrix that allows to represent the rotated system per each element in the reference system. This matrix is the following:

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Knowing that $a = 1$, $A^1 = A^2 = A$ and $A^3 = A'$ the elemental stiffness matrix for each element can be computed.

Element 1:

$$K^1 = EA \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{16}$$

Element 2:

$$K^2 = EA \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Element 3:

$$K^3 = \frac{EA'}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{18}$$

Building the matrix for the global system:

$$K = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & 0 \\ -A & 0 & A+\frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} \\ 0 & 0 & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} \\ 0 & 0 & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} \\ 0 & 0 & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & A+\frac{\sqrt{2}A'}{4} \end{bmatrix} \tag{19}$$

b)

Even changing the values for cross sections of A_1, A_2, A_3, it won't be a set of values that makes the stiffness matrices of the two models equals to each other. From a mathematical point of view, this happens because there isn't any linear combination between the two matrices. Even changing the area ratio between the bars in the second model, the two matrices will be different. The value that makes the two matrices the most similar, is $A = \frac{3}{4}$ and $A' = \frac{1}{\sqrt{2}}$. So the terms on diagonal will be almost the same, even if it is impossible to equalize the other terms.

c)

This makes sense, because from a physical point of view the two systems are not the same. The first one is a unique structure and if we want to look at it as an assembly of three elements, they are linked with three interlocking, which holds stresses in all the directions and transmit also the rotation. On the other hand, the second model is structure made by three elements linked by three hinges. The hinges, different from the interlocking, doesn't hold the rotation.

Moreover, as can be noticed by the methods applied in order to compute both the matrices, there is a difference in the loads that each structure can handle and how the boundary conditions are applied. In the Turner triangle, the loads are applied in the whole structure and the whole structure reacts to the loads. In the second model, the BCs are applied only on the nodes, and the axial directions are the ones that reacts to these loads.

d)

The Poisson coefficient ν describes how the material reacts to the stresses. When $\nu = 0$, the stresses are described by a diagonal matrix. It means that the stress in one direction, are caused only by a strain in the same direction. When $\nu \neq 0$, the stress in one direction is generated by displacements in different directions as well. So the matrix 10 describes these materials behaviours depending on the Poisson values. (It can be noticed that if $\nu = 0$ the 10 will be diagonal).

In the case of $\nu \neq 0$, the stiffness matrix of the first model is:

$$K = \frac{E}{2(1-\nu^2)} \begin{bmatrix} 1+\frac{1-\nu}{2} & \nu+\frac{1-\nu}{2} & -1 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\nu \\ \nu+\frac{1-\nu}{2} & 1+\frac{1-\nu}{2} & -\nu & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

It can be noticed that, in the non-diagonal terms, it appears this ν value that is the stress contribution caused by the strain in other directions. It is evident in the terms $K_{12}, K_{23}, K_{16}, K_{36}$, and in the respective ones due to the symmetry of the matrix, that these describes the behaviour of generating stress in directions different from the displacements one.

Moreover, the stiffness matrix is derived from the MPE principle, so from the energy balance. When $\nu = 0$, the strains along x and y are zero, so the total internal energy in the body will be smaller than in the case if $\nu \neq 0$. In fact, the terms of the stiffness matrix, considering $\nu = 0$, are smaller than the ones in the other case. As can be noticed, the terms of 20 differ from the ones in 13 by a $-\nu$ divided by terms of $-\nu^2$ and considering that $\nu < 1$, this will make the terms greater. This match exactly with the physical meaning.

In the second model, it has been used the Direct Stiffness Method (DSM) and the ν coefficient has not been involved. The bars have been considered linear and the physical meaning is maintained by the Young Modulus and the section of each bars. The stiffness matrix of the second model won't be affected by the Poisson ratio.