Master's Degree Numerical Methods in Engineering

Universitat Politècnica de Catalunya BarcelonaTech

Computational Structural Mechanics and Dynamics

Assignment 4: Isoparametric representation

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1 Assignment 4.1

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x_1 - x_2$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacement u1, u2 and u3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

in which $N_1^e(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint x = l/2. For convenience define,

$$x_1 = 0, x_2 = l, x_3 = (1/2 + \alpha)l$$

where $-1/2 < \alpha < 1/2$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

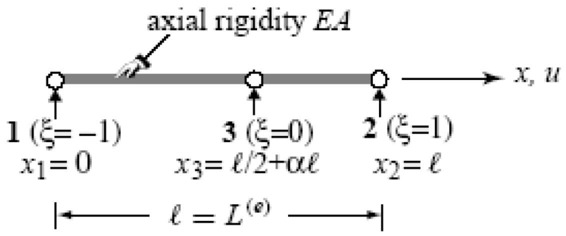

Figure 1: Fig.1 The three-node bar element in its local system

1. From (7.2) and the second equation of (7.1) get the Jacobian $J = dx/d\xi$ in terms of l, α and ξ. Show that,

if $-1/4 < \alpha < 1/4$ then $J > 0$ over the whole element $-1 < \xi < 1$
if $\alpha = 0$, $J = l/2$ is a constant over the element.

2. Obtain the 1x3 strain displacement matrix B relating $e = du/dx = Bu_e$ where u_e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of B are functions of l, α and ξ.

1. In the first step we have to build up the shape functions for the quadratic 1D element in terms of ξ :

$$N_1 = \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)}$$

$$N_1 = \frac{(\xi_3 - \xi)(\xi_1 - \xi)}{(\xi_3 - \xi_2)(\xi_1 - \xi_2)}$$

$$N_1 = \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)}$$

Solving the N for $\xi_1 = -1, \xi_2 = 1, \xi_3 = 0$ we will have:

$$N_1 = \xi^2 - \xi/2, N_2 = \xi^2 + \xi/2, N_3 = 1 - \xi^2$$

We know that:

$$x = x_1 dN_1/d\xi + x_2 dN_2/d\xi + x_3 dN_3/d\xi$$

Using the given values for the x coordinates of the nodes in the physical world. The jacobian will be the sum of the derivatives of the shape functions in ξ

$$J = 0(\xi - 1/2) + l(\xi + 1/2) + (l/2 + l\alpha)(-2\xi)$$

$$J = \frac{l(1 - 4\alpha\xi)}{2}$$

Considering the case where $-1/4 < \alpha < 1/4$, from the calculated jacobian we can see that for this cases we will have:

$$\alpha = -1/4 : J = l(1 + \xi)/2$$

$$\alpha = 1/4 : J = l(1-\xi)/2$$

Where both cases are positive because $-1 < \xi < 1$. So the minimum for the jacobian in both cases will be zero. For the case of $\alpha = 0$ we see that then the jacobian will be $J = l/2$ which is a constant.

2. For building up the B matrix we have to calculate J^{-1}. In this case because the J is a one by one matrix so the inverse will be the inverse of the jacobian number. So the $J^{-1} = 2/l(1 - 4\alpha\xi)$ and we can calculate the $B = J^{-1}dN/d\xi$.

$$B = 2/l(1-4\alpha\xi)\begin{bmatrix}\xi - 1/2 & \xi + 1/2 & -2\xi\end{bmatrix}$$

2 Assignment 4.2

1. Compute the entries of k^e for the following axisymmetric triangle:

$$r_1 = 0, r_2 = r_3 = a, z_1 = z_2 = 0, z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$E = E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of k^e must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector f^e for gravity forces $b = [0, -g]^T$.

1. For computing the stiffness matrix of this axisymmetric element we first calculate the needed coefficients of the element as:

$$a_1 = ab, b_1 = -b, c_1 = 0$$

$$a_2 = 0, b_2 = b, c_1 = -a$$

$$a_3 = 0, b_3 = 0, c_3 = a$$

And the area of the element is $A = ab/2$. After this stage we calculate the shape functions using the formula below:

$$N_i = a_i + b_i r + c_i z/2A$$

So the shape functions will be:

$$N_1 = 1 - r/a, N_2 = r/a - z/b, N_3 = z/b$$

We calculate the stiffness matrix using:

$$K_{ij}^e = 2\pi B_i^T D B_j \bar{r} A$$

Where the D matrix is the E matrix given, $\bar{r} = r_1 + r_2 + r_3/3 = 2a/3$, the A is the area of the element and the B matrices should be calculated for each point using the shape function as below:

$$B_i = \begin{bmatrix} \partial N_i/\partial r & 0 \\ 0 & \partial N_i/\partial z \\ N_i/r & 0 \\ \partial N_i/\partial r & \partial N_i/\partial z \end{bmatrix}$$

The general B matrix will be calculated as:

$$B = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ 1-1/a & 0 & 1/a - z/b & 0 & z/br & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

Where every two column is the B_i of the associated node. Calculating every K_{ij} and assembling the main stiffness matrix we will have:

$$K = 2\pi a^2 bE/3$$

$$\begin{bmatrix} 1/a^2 + (1-1/a)^2 & 0 & -2/a^2 + (b+2a+z)/ab & 0 & z/br(1-1/a) & 0 \\ & 1/2a^2 & 1/2ab & -1/2a^2 & -1/2ab & 0 \\ & & 1/a^2 + (1/a - z/b)^2 + 1/2b^2 & -1/2ab & z/br(1/a - z/b) - 1/2b^2 & 0 \\ & & & 1/b^2 + 1/2a^2 & 1/2ab & -1/b^2 \\ & & & & (z/br)^2 + 1/2b^2 & 0 \\ & symmetric & & & & 1/b^2 \end{bmatrix}$$

2. Summing the 2,4 and 6 row we can see that they vanish but the same does not happen for the other rows (because the stiffness matrix is symmetric then the same applies for the associated columns). For example summing for the second row we can see:

$$1/2a^2 + 1/2ab - 1/2a^2 - 1/2ab = 0$$

The 2,4 and 6 rows represent the z-direction and the others the r-direction. The reason that the sum of this rows is zero is that, because we have a axisymmetric model, we are free to have rigid body movments on the z-direction in which case because we wont have any strains in the element then there will be not stresses and in conclusion no forces on the element. But in the case of the r-direction because of the axisymmetric model it is not allowed to have rigid body movements so the terms do not cancel out each other.

3. The body forces for each element is calculated as:

$$f_i^e = -2\pi \begin{pmatrix} b_r \\ b_z \end{pmatrix} \bar{r}A/3$$

So the general body force matrix will be calculated as:

$$f^e = 2\pi(ab)^2/9 \begin{bmatrix} 0 \\ g \\ 0 \\ g \\ 0 \\ g \end{bmatrix}$$

3 References

Zienkiewicz - The Finite Element Method I
CSMD-06-Isoparametric
CSMD-07-Revolution