UPC - BARCELONA TECH
MSc Computational Mechanics
Spring 2018

Coputations Solid Mechanics & Dynamics

Assignment 4

Due 5/03/2018

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"Structures of revolution"

Assignment 4.1

1. Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\mathbf{f}^e$ for gravity forces $\mathbf{b} = [0, -g]^T$.

Date of Assignment: 26 / 02 / 2018
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The assignment must be submitted as a pdf file named **As4-Surname.pdf** to the CIMNE virtual center.

4.1 Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle

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The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ r \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix} \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix} = Nu^e$$

The shape function for linear triangle are precisely the triangular (area) coordinates
$N_1^e = \zeta_1, N_2^e = \zeta_2, N_3^e = \zeta_3$
From (1), we get,
$r = r_1 N_1^e + r_2 N_2^e + r_3 N_3^e = r_1\zeta_1 + r_2\zeta_2 + r_3\zeta_3$

But we know that $\text{r}_1 = 0 and r_2 = r_3 = a$,
$\therefore r = a(\zeta_2 + \zeta_3)$
$z = z_1 N_1^e + z_2 N_2^e + z_3 N_3^e = z_1\zeta_1 + z_2\zeta_2 + z_3\zeta_3$
$z_1 = z_2 = 0, z_3 = a$
$\therefore z = b\zeta_3$

Thus, we can find the shape functions as follows, $z = b\zeta_3$

$$\boxed{\therefore \zeta_3 = N_3^e = \frac{z}{b}}$$

$r = a(\zeta_2 + \zeta_3) = a(\zeta_2 + \frac{z}{b})$

$$\boxed{\therefore \zeta_2 = N_2^e = \frac{r}{a} - \frac{z}{b}}$$

We know that for any element, $N_1^e + N_2^e + N_3^e = 1$
$N_1^e = 1 - N_2^e - N_3^e = 1 - \frac{r}{a} + \frac{z}{b} - \frac{z}{b}$

$$\boxed{\therefore \zeta_1 = N_1^e = 1 - \frac{r}{a}}$$

The strain matrix of element is

$$\mathbf{B}^e = \left[\begin{array}{cc:cc:cc} \frac{\partial N_1^e}{\partial r} & 0 & \frac{\partial N_2^e}{\partial r} & 0 & \frac{\partial N_3^e}{\partial r} & 0 \\ 0 & \frac{\partial N_1^e}{\partial z} & 0 & \frac{\partial N_2^e}{\partial z} & 0 & \frac{\partial N_3^e}{\partial z} \\ \frac{N_1^e}{r} & 0 & \frac{N_2^e}{r} & 0 & \frac{N_3^e}{r} & 0 \\ \frac{\partial N_1^e}{\partial z} & \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_2^e}{\partial r} & \frac{\partial N_3^e}{\partial z} & \frac{\partial N_3^e}{\partial r} \end{array}\right]$$

$$\mathbf{B}^e = \begin{bmatrix} \mathbf{B}_1^e & \mathbf{B}_2^e & \mathbf{B}_3^e \end{bmatrix}$$

Calculating the partial derivatives for shape functions w.r.t r and z

$$\frac{\partial N_1^e}{\partial r} = \frac{-1}{a}, \frac{\partial N_2^e}{\partial r} = \frac{1}{a}, \frac{\partial N_1^e}{\partial r} = 0$$
$$\frac{\partial N_1^e}{\partial z} = 0, \frac{\partial N_1^e}{\partial z} = \frac{-1}{b}, \frac{\partial N_1^e}{\partial z} = \frac{1}{b}$$

$$\mathbf{B}^e = \left[\begin{array}{cc:cc:cc} \frac{-1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-b} & 0 & \frac{1}{b} \\ \frac{1}{r} - \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{br} & 0 & \frac{z}{br} & 0 \\ 0 & \frac{-1}{a} & \frac{-1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{array}\right]$$

Element stiffness matrix $\mathbf{K}^e = \int_{\Omega^e} r\mathbf{B}^e\mathbf{E}\mathbf{B}d\Omega^e$
But First we will calculate the term r $\mathbf{B}^e\mathbf{E}\mathbf{B}$ and integrate the result in next step.

$$r\mathbf{B}^e\mathbf{E}\mathbf{B} = r \begin{bmatrix} \frac{-1}{a} & 0 & \frac{1}{r} - \frac{1}{a} & 0 \\ 0 & 0 & 0 & \frac{-1}{a} \\ \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{br} & \frac{-1}{b} \\ 0 & \frac{-1}{b} & 0 & \frac{1}{a} \\ 0 & 0 & \frac{z}{br} & \frac{1}{b} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-b} & 0 & \frac{1}{b} \\ \frac{1}{r} - \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{br} & 0 & \frac{z}{br} & 0 \\ 0 & \frac{-1}{a} & \frac{-1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$
r\mathbf{B}^e\mathbf{E}\mathbf{B} = E \begin{bmatrix} \frac{-r}{a} & 0 & 1-\frac{r}{a} & 0 \\ 0 & 0 & 0 & \frac{-r}{a} \\ \frac{r}{a} & 0 & \frac{r}{a}-\frac{z}{b} & \frac{-r}{b} \\ 0 & \frac{-r}{b} & 0 & \frac{r}{a} \\ 0 & 0 & \frac{z}{b} & \frac{r}{b} \\ 0 & \frac{r}{b} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-b} & 0 & \frac{1}{b} \\ \frac{1}{r}-\frac{1}{a} & 0 & \frac{1}{a}-\frac{z}{br} & 0 & \frac{z}{br} & 0 \\ 0 & \frac{-1}{a} & \frac{-1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}
$$

After completing all Matrix multiplication we get the following matrix

$$
r\mathbf{B}^e\mathbf{E}\mathbf{B} = E \left[\begin{array}{c:c:c:c:c:c} \frac{2r}{a^2}-\frac{2}{a}+\frac{1}{r} & 0 & 0 & 0 & \frac{z}{br}(1-\frac{r}{a}) & 0 \\ 0 & \frac{r}{2a^2} & \frac{r}{2ab} & \frac{-r}{2a^2} & \frac{-r}{2ab} & 0 \\ 0 & \frac{r}{2ab} & (\frac{2r}{a^2}-\frac{2z}{ab}+\frac{z^2}{b^2r}-\frac{r}{2b^2}) & \frac{-r}{2ab} & (\frac{z}{ab}-\frac{z^2}{rb^2}-\frac{r}{2b^2}) & 0 \\ 0 & \frac{-r}{2a^2} & \frac{-r}{2ab} & (\frac{r}{b^2}+\frac{r}{2a^2}) & \frac{r}{2ab} & \frac{-r}{b^2} \\ \frac{z}{br}(1-\frac{r}{a}) & \frac{-r}{2ab} & (\frac{z}{ab}-\frac{z^2}{b^2r}-\frac{r}{2b^2}) & \frac{r}{2ab} & (\frac{z^2}{rb^2}+\frac{r}{2b^2}) & 0 \\ 0 & 0 & 0 & \frac{-r}{b^2} & 0 & \frac{r}{b^2} \end{array}\right]
$$

In order to lately integrate this result using Guass rule, we need to have this matrix as a function of the function of the area coordinates $\zeta_1, \zeta_2, \zeta_3$. We can obtain this result by substituting $r = a(\zeta_1 + \zeta_3)$ and $z = b\zeta_3$. $\therefore \mathbf{K}^e = \int_{\Omega^e} r\mathbf{B}^e\mathbf{E}\mathbf{B}d\Omega^e =$

Please turn over

$$
\mathbf{K}^e = \int_\Omega \left[\begin{array}{c:c:c:c:c:c}
\frac{2}{a}(\zeta_2+\zeta_3-1)+\frac{2}{a(\zeta_2+\zeta_3)} & 0 & 0 & 0 & \frac{\zeta_3}{a(\zeta_2+\zeta_3)}(1-\zeta_2-\zeta_3) & 0 \\
0 & \frac{(\zeta_2+\zeta_3)}{2a} & \frac{(\zeta_2+\zeta_3)}{2b} & \frac{-(\zeta_2+\zeta_3)}{2a^2} & \frac{-(\zeta_2+\zeta_3)}{2ab} & 0 \\
0 & \frac{(\zeta_2+\zeta_3)}{2b} & \frac{1}{a}(2(\zeta_2+\zeta_3)-2+\frac{\zeta^2}{\zeta_2+\zeta_3}) & -\frac{(\zeta_2+\zeta_3)}{2b} & (\frac{\zeta_3}{a}-\frac{\zeta^2}{a(\zeta_2+\zeta_3)}-\frac{a(\zeta_2+\zeta_3)}{2b^2}) & 0 \\
0 & \frac{-(\zeta_2+\zeta_3)}{2a} & \frac{-(\zeta_2+\zeta_3)}{2b} & (\frac{a(\zeta_2+\zeta_3)}{b^2}+\frac{(\zeta_2+\zeta_3)}{2a}) & \frac{(\zeta_2+\zeta_3)}{2b} & \frac{-a(\zeta_2+\zeta_3)}{b^2} \\
\frac{\zeta_3}{a(\zeta_2+\zeta_3)}(1-\zeta_2-\zeta_3) & \frac{-(\zeta_2+\zeta_3)}{2b} & (\frac{\zeta_3}{a}-\frac{\zeta^2}{a(\zeta_2+\zeta_3)}-\frac{a(\zeta_2+\zeta_3)}{2b^2}) & \frac{(\zeta_2+\zeta_3)}{2b} & (\frac{\zeta_3^2}{a(\zeta_2+\zeta_3)}+\frac{a(\zeta_2+\zeta_3)}{2b^2}) & 0 \\
0 & 0 & 0 & \frac{-a(\zeta_2+\zeta_3)}{b^2} & 0 & \frac{a(\zeta_2+\zeta_3)}{b^2}
\end{array}\right] d\Omega
$$

For simplicity we can now use the centroid Guass rule (1point, degree 1) to get a suitable approximation of the $\mathbf{K}^e$ Centroid rule $\Rightarrow \frac{1}{A}\int_\Omega F(\zeta_1,\zeta_2,\zeta_3)d\Omega \approx F(\frac{1}{3},\frac{1}{3},\frac{1}{3}), A = \frac{ab}{2}$

$$
\boxed{\mathbf{K}^e = \frac{ab}{2}\left[\begin{array}{cc:cc:cc}
\frac{5}{6a} & 0 & 0 & 0 & \frac{1}{6a} & 0 \\
0 & \frac{1}{3a} & \frac{1}{3b} & \frac{-1}{3a} & \frac{-1}{3b} & 0 \\
0 & \frac{1}{3b} & \frac{5}{6a}-\frac{a}{3b^2} & \frac{-1}{3b} & \frac{1}{6a}-\frac{a}{3b^2} & 0 \\
0 & \frac{-1}{3a} & \frac{-1}{3b} & \frac{2a}{3b^2}+\frac{1}{3a} & \frac{-1}{3b} & \frac{-2a}{3b^2} \\
\frac{1}{6a} & \frac{-1}{3b} & \frac{1}{6a}-\frac{a}{3b^2} & \frac{1}{3b} & \frac{1}{6a}+\frac{a}{3b^2} & 0 \\
0 & 0 & 0 & \frac{-2a}{3b^2} & 0 & \frac{2a}{3b^2}
\end{array}\right]}
$$

4.2 The Sum of row(column 2,4,6)

For row 2
$\sum_{i=1}^{6} K_{i2} = 0 + \frac{1}{3a} + \frac{1}{3b} + \frac{-1}{3a} + \frac{-1}{3b} + 0 = 0$
For Row 4
$\sum_{i=1}^{6} K_{i4} = 0 + \frac{-1}{3a} + \frac{-1}{3b} + \frac{2a}{3b^2} + \frac{1}{3a} + \frac{-1}{3b} + \frac{-2a}{3b^2} = 0$
For row 6
$\sum_{i=1}^{6} K_{i6} = 0 + 0 + 0 + \frac{-2a}{3b^2} + 0 + \frac{2a}{3b^2} = 0$

For row 1
$\sum_{i=1}^{6} K_{i1} = \frac{5}{6a} + \frac{1}{6a} \neq 0$
For Row 3
$\sum_{i=1}^{6} K_{i3} = \frac{1}{3b} + \frac{5}{6a} - \frac{a}{3b^2} - \frac{1}{3b} + \frac{1}{6a} - \frac{a}{3b^2} = \frac{1}{a} - \frac{2a}{3b^2} \neq 0$
For row 5
$\sum_{i=1}^{6} K_{i5} = \frac{1}{6a} - \frac{1}{3b} + \frac{1}{6a} - \frac{a}{3b^2} + \frac{1}{3b} + \frac{1}{6a} + \frac{a}{3b^2} = \frac{1}{2a}$

In the way that the vector of displacements $\underline{u}^e$ was defined as

$$\underline{u}^e = \begin{bmatrix} u_{r1} & u_{z1} & u_{r2} & u_{z2} & u_{r3} & u_{z3} \end{bmatrix}^T$$

Thus, on a linear system Ku = , rows(columns) $2, 4, 6$ corresponds to stiffness affecting the vertical displacements while rows (columns) $1, 3, 5$ affect radial displacements.
A noteworthy aspect of structures of revolution is the appearance of the "hoop" strain $e_{\theta\theta} = u_r/r$. A uniform radial displacement is no longer a rigid body motion. Instead, it produces a circumferential strain.
When rows(columns) of a stiffness matrix sums to zero, it means that there is no internal energy associated with a particular degree of freedom. Thus, the stiffness matrix is "able to reproduce" solid rigid motions.
According to this reasoning, the columns (rows) of the matrix $\underline{K}^e$ corresponding to radial displacements should not sum to zero (because in revolution structures we have the hoop strain). On the contrary, rows $2, 4 and 6$ must vanish since for this structures we have one possible rigid body motion, the vertical one.

4.3 Computing the consistent force vector $\mathbf{f}^e$ for gravity force $\underline{\mathbf{b}} = [0, -g]^T$

$$\mathbf{N} = \begin{bmatrix} N_e^1 & 0 & N_e^2 & 0 & N_e^3 & 0 \\ 0 & N_e^1 & 0 & N_e^2 & 0 & N_e^3 \end{bmatrix}$$

$\underline{f}^e = \int_\Omega \mathbf{N}^T \underline{b} r d\Omega$,
$N_e^1 = \zeta_1, N_e^2 = \zeta_2, N_e^3 = \zeta_3$

$$\underline{f}^e = \int_\Omega \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r d\Omega = \begin{bmatrix} 0 \\ -\zeta_1 g r \\ 0 \\ -\zeta_2 g r \\ 0 \\ -\zeta_3 g r \end{bmatrix} d\Omega$$

But $r = a(\zeta_2 + \zeta_3)$, then

$$\underline{f}^e = \int_\Omega \begin{bmatrix} 0 \\ -\zeta_1 ga(\zeta_2+\zeta_3) \\ 0 \\ -\zeta_2 ga(\zeta_2+\zeta_3) \\ 0 \\ -\zeta_3 ga(\zeta_2+\zeta_3) \end{bmatrix} d\Omega = \int_\Omega F(\zeta_1, \zeta_2, \zeta_3) d\Omega$$

And as the polynomial function to integrate is of degree 2, we need a 3 point Guass quadrature. We can use, for instance, the midpoint rule for a straight side triangle.

$$\underline{f}^e \cong A[\frac{1}{3}F(\frac{1}{2}), \frac{1}{2}, 0) + \frac{1}{3}F(0, \frac{1}{2}, \frac{1}{2}) + \frac{1}{3}F(\frac{1}{2}, 0, \frac{1}{2})]$$

$$F(\tfrac{1}{2}), \tfrac{1}{2}, 0) = \begin{bmatrix} 0 \\ -\frac{ga}{4} \\ 0 \\ -\frac{ga}{4} \\ 0 \\ 0 \end{bmatrix} \quad F(0, \tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{ga}{2} \\ 0 \\ -\frac{ga}{2} \end{bmatrix} \quad F(\tfrac{1}{2}, 0, \tfrac{1}{2}) = \begin{bmatrix} 0 \\ -\frac{ga}{4} \\ 0 \\ 0 \\ 0 \\ -\frac{ga}{4} \end{bmatrix}$$

Finally it yields,

$$\underline{f}^e = \frac{ab}{2} \begin{bmatrix} 0 \\ -\frac{ga}{6} \\ 0 \\ -\frac{ga}{4} \\ 0 \\ -\frac{ga}{4} \end{bmatrix} = \boxed{\frac{-a^2bg}{4} \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}}$$