Computational Structural Mechanics and Dynamics, Assignment 4

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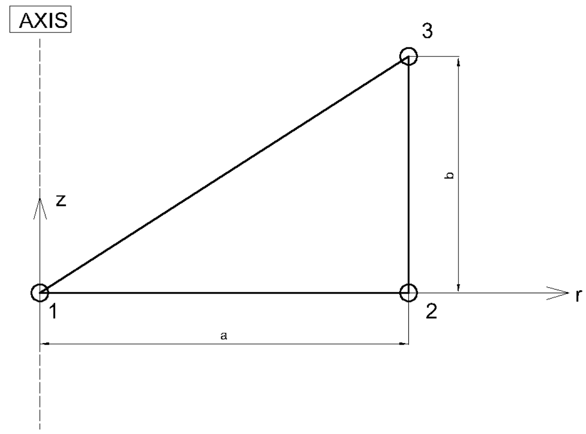
Assignment 4.1
On "Structures of Revolution":

1. Compute the entries of $\boldsymbol{K}^e$ for the following axisymmetric triangle:

$$r1 = 0, \quad r2 = r3 = a, \quad z1 = z2 = 0, \quad z3 = b$$

The material is isotropic with $\nu = 0$ for which the strain-stress matrix is:

$$\boldsymbol{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$



In order to calculate the stiffness matrix, one can start by selecting the shape functions arrangement (we will use classical linear shape functions ξ, η, $1-(\xi-\eta)$):

$$\boldsymbol{N} = \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 \end{bmatrix}$$

And the vector of unknowns:

$$\boldsymbol{u} = \begin{bmatrix} r1 & z1 & r2 & z2 & r3 & z3 \end{bmatrix}^T$$

Following the derivation one arrives to the following $\mathbf{B}$ matrix:

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial N1}{\partial r} & 0 & \frac{\partial N2}{\partial r} & 0 & \frac{\partial N3}{\partial r} & 0 \\ 0 & \frac{\partial N1}{\partial z} & 0 & \frac{\partial N2}{\partial z} & 0 & \frac{\partial N3}{\partial z} \\ \frac{N1}{r} & 0 & \frac{N2}{r} & 0 & \frac{N3}{r} & 0 \\ \frac{\partial N1}{\partial z} & \frac{\partial N1}{\partial r} & \frac{\partial N2}{\partial z} & \frac{\partial N2}{\partial r} & \frac{\partial N3}{\partial z} & \frac{\partial N3}{\partial r} \end{bmatrix}$$

One can observe the difficulty of dealing with a term involving the geometry in the discretization of the equations. Taking one Gauss point, the approximation for the $\mathbf{B}$ matrix is:

$$\boldsymbol{B} \approx \begin{bmatrix} \frac{y_{23}}{2A} & 0 & \frac{y_{31}}{2A} & 0 & \frac{y_{12}}{2A} & 0 \\ 0 & \frac{x_{32}}{2A} & 0 & \frac{x_{13}}{2A} & 0 & \frac{x_{21}}{2A} \\ \frac{1}{3\bar{r}} & 0 & \frac{1}{3\bar{r}} & 0 & \frac{1}{3\bar{r}} & 0 \\ \frac{x_{32}}{2A} & \frac{y_{23}}{2A} & \frac{x_{13}}{2A} & \frac{y_{31}}{2A} & \frac{x_{21}}{2A} & \frac{y_{12}}{2A} \end{bmatrix}$$

Where $\bar{r} = (r1 + r2 + r3)/3$. Appriximating then **K** as the evaluation at the baricenter of the triangle of :

$$\boldsymbol{K} \approx \bar{r} A \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B}$$

One gets:

$$\boldsymbol{K} = E \begin{bmatrix} \frac{4ba^4}{243} + \frac{b}{3} & 0 & \frac{4ba^4}{243} - \frac{b}{3} & 0 & \frac{4ba^4}{243} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ \frac{4ba^4}{243} - \frac{b}{3} & \frac{a}{6} & \frac{4ba^4}{243} + \frac{b}{3} + \frac{a^2}{6b} & -\frac{a}{6} & \frac{4ba^4}{243} - \frac{a^2}{6b} & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{a^2}{3b} + \frac{b}{6} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{4ba^4}{243} & -\frac{a}{6} & \frac{4ba^4}{243} - \frac{a^2}{6b} & \frac{a}{6} & \frac{4ba^4}{243} + \frac{a^2}{6b} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

2.Show that the Show that the sum of the rows (and columns) 2, 4 and 6 of $\boldsymbol{K}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

One can easily observe from $\boldsymbol{K}$ that the sum of the even columns is null. This is due to the fact that, as stated in the beginning of the report, the even indices of the unknowns vector are related to the z coordinate, whose variations can be related to rigid body motions, that is, they do not consume energy.

Likewise, one can note that the odd indices of the vector of unknowns are related to changes on the r coordinate. Since in the axisymmetric modelation on is considering a ring of material, it is possible to say that changes in the r coordinate will consume energy.

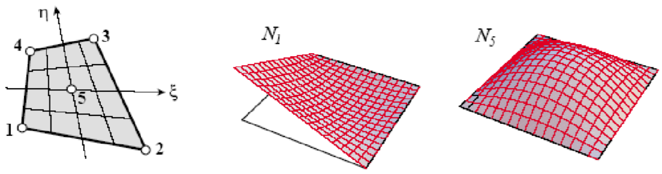
3. Compute the consistent force vector $\boldsymbol{f}$ for gravity forces $\boldsymbol{b}$ = [0, –g]T.

One can also use one Gauss point at the baricenter of the element to calculate the forcing term as:

$$\boldsymbol{f} \approx \frac{\bar{r}A}{3}[0, -g, 0, -g, 0, -g]^T = \frac{a^2 b}{9}[0, -g, 0, -g, 0, -g]^T$$

Assignment 4.2
On "Isoparametric Representation":

A five node quadrilateral element has the nodal configuration shown if the figure. Perspective views of N1 and N5 are shown in the same figure.



Find five shape functions Nei, i=1,...,5 that satisfy compatibility and also verify that their sum is unity.

One can start by obtaining the shape function of the node number 5 (at the center of the element), and it is the following:

$$N5 = (1-\eta^2)(1-\xi^2)$$

Now, the expression of the shape functions for a linear quadrilateral are readily given as:

$$\hat{N1} = \frac{1}{4}(1-\xi)(1-\eta) \quad \hat{N2} = \frac{1}{4}(1+\xi)(1-\eta) \quad \hat{N3} = \frac{1}{4}(1+\xi)(1+\eta) \quad \hat{N4} = \frac{1}{4}(1-\xi)(1+\eta)$$

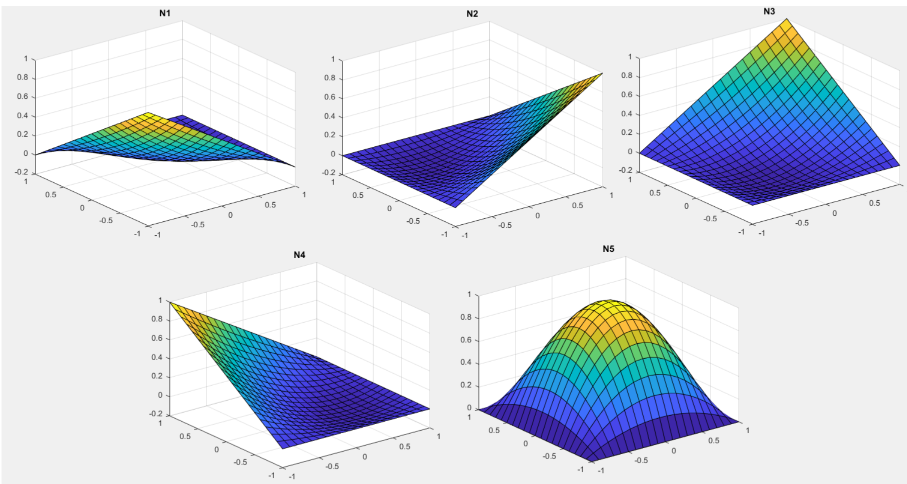
One now only need to find the correct combination of them by solving the following equation:

$$N_i = \hat{N}_i + \alpha N5 \qquad \alpha = -\frac{1}{4}$$

And so:

$$N_i = \hat{N}_i - \frac{1}{4} N5$$

A MATLAB script was created to visualize the shape functions. The resulting plots are shown next:



Likewise at every point of the grid N1+N2+N3+N4+N5 = 1 .