Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Computational Solid Mechanics and Dynamics

Isoparametric Representation
Structures of Revolution

Assignment 4

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March 9, 2020

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1 Assignment 4.1

1.1 Statement

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x1 - x2$. The axial displacement is $u(x)$. The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \tag{1}$$

in which $N_1^e 1(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = l/2$. For convenience define,

$$x_1 = 0 \qquad x_2 = l \qquad x_3 = \left(\tfrac{1}{2} + \alpha\right) l \tag{2}$$

where $-0.5 < \alpha < 0.5$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

Question 1. From equation 3 and the second equation of 1 get the Jacobian $J = dx/\,d\xi$ in terms of l, α and ξ. Show that,

- if $1/4 < \alpha < 1/4$ then $J > 0$ over the whole element $-1 < \xi < 1$
- if $\alpha = 0$, $J = 1/2$ is a constant over the element.

Question 2. Obtain the 1x3 strain displacement matrix $\boldsymbol{B}$ relating $e = du/dx = Bu^e$ where u^e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of $\boldsymbol{B}$ are functions of l, α and ξ.

1.2 Solution

Question 1 We will start by defining the isoparamentric element:

$$\xi_1 = 0 \qquad \xi_2 = 1 \qquad \xi_3 = \tfrac{1}{2} + \alpha \tag{3}$$

The transformation must follow:

$$x = A\xi^2 + B\xi + C \tag{4}$$

Then we have the following system of equations:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ (1/2 + \alpha)l \\ l \end{bmatrix} \tag{5}$$

Solving and substituting yields:

$$x = (-2\alpha\xi^2 + \xi + 1 + 2\alpha)\frac{l}{2} \tag{6}$$

Obtaining the jacobian becomes very simple:

$$J = \frac{dx}{d\xi} = \frac{l}{2}(1 - 4\alpha\xi) \tag{7}$$

We must now proof that $J > 0$ in $\alpha \in (-\frac{1}{4}, \frac{1}{4})$. It is a linear equation so checking that the limits are positive (non-strictly, since its an open interval) is sufficient. More formaly:

$$J(\alpha) > 0, \quad \alpha \in (\alpha_0, \alpha_1) \iff J(\alpha_0), J(\alpha_1) \geq 0 \tag{8}$$

where α_0 and α_1 are $\pm\frac{1}{4}$ respectively. Evaluating at this limits yields:

$$\begin{aligned} J\Big|_{\alpha=-\frac{1}{4}} &= \frac{l}{2}(1+\xi) \\ J\Big|_{\alpha=+\frac{1}{4}} &= \frac{l}{2}(1-\xi) \end{aligned} \tag{9}$$

Since $\xi \in [-1, +1]$, we can check that both previous expressions are 0 in the worst case, positive in all others. As said before, the inequality needs not be strict so we have confirmed that $J > 0$.

Moving on to the following assertion, it says that for $\alpha = 0$, J is half the length in all domain. Let's start by substituting in equation 7:

$$J = \frac{l}{2}(1 - 4\alpha\xi) = \frac{l}{2} \tag{10}$$

Question 2. We must first define our shape functions:

$$\left.\begin{aligned} N_1(\xi) &= -\frac{1}{2}\xi(1-\xi) \\ N_2(\xi) &= +\frac{1}{2}\xi(1-\xi) \\ N_3(\xi) &= 1 - \xi^2 \end{aligned}\right\} \tag{11}$$

Let's now compute their derivatives:

$$\left.\begin{aligned} \tfrac{dN_1}{d\xi} &= \xi - \frac{1}{2} \\ \tfrac{dN_2}{d\xi} &= \frac{1}{2} - \xi \\ \tfrac{dN_3}{d\xi} &= -2\xi \end{aligned}\right\} \tag{12}$$

We only need the inverse of the jacobian. Recalling equation 7 we have that:

$$J^{-1} = \frac{2}{l(1 - 4\alpha\xi)} \tag{13}$$

Then matrix $\boldsymbol{B}$ is:

$$\boldsymbol{B} = \frac{2}{l(1-4\alpha\xi)} \begin{bmatrix} \xi - \frac{1}{2} \\ \frac{1}{2} - \xi \\ -2\xi \end{bmatrix}^T \tag{14}$$

2 Assignment 4.2

2.1 Statement

Question 1. Compute the entries of $\boldsymbol{K}^e$ for the following axisymmetric triangle:

$$\begin{matrix} r_1 = 0 & r_2 = a & r_3 = a \\ z_1 = 0 & z_2 = 0 & z_3 = b \end{matrix}$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{15}$$

Question 2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\boldsymbol{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

Question 3. Compute the consistent force vector $\boldsymbol{f}^e$ for gravity forces $b = [0, -g]^T$.

2.2 Solution

Question 1. To compute the stiffness matrix we'll use the following expression:

$$\boldsymbol{K}^e = \int_{\Omega^3} \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} \, dV = 2\pi \int_{\Omega^2} \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} r \, dS \tag{16}$$

where Ω^3 is the whole 3D domain and Ω^2 is the 2D cross-sectional simplified domain. In order to obtain $\boldsymbol{B}$ we must first define the shape functions:

$$\left. \begin{aligned} N_1(r,z) &= 1 - \frac{r}{a} \\ N_2(r,z) &= \frac{r}{a} - \frac{z}{b} \\ N_3(r,z) &= \frac{z}{b} \end{aligned} \right\} \tag{17}$$

Now we can obtain matrix B according to its definition:

$$\boldsymbol{B} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3] \quad \text{where} \quad \boldsymbol{B}_i = \begin{bmatrix} \frac{dN_i}{dr} & 0 \\ 0 & \frac{dN_i}{dz} \\ \frac{N_i}{r} & 0 \\ \frac{dN_i}{dz} & \frac{dN_i}{dr} \end{bmatrix} \tag{18}$$

It can be seen that it is a function of r and z. To avoid over-complicating the integral in equation 17 we can approximate it by evaluating it at the barycenter $r_c = \frac{1}{3}[2a, b]$. This yields:

$$\boldsymbol{B} = \left[\begin{array}{cc|cc|cc} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{array}\right] \tag{19}$$

We can now transform equation 17:

$$\boldsymbol{K}^e = 2\pi \int_{\Omega^2} \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} r \, dS = 2\pi \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} r_c S \tag{20}$$

Evaluating this becomes:

$$\boldsymbol{K} = \frac{\pi E}{6ab} \begin{bmatrix} 5ab^2 & 0 & -3ab^2 & 0 & ab^2 & 0 \\ 0 & 2ab^2 & 2a^2b & -2ab^2 & -2a^2b & 0 \\ -3ab^2 & 2a^2b & 5ab^2+2a^3 & -2a^2b & ab^2-2a^3 & 0 \\ 0 & -2ab^2 & -2a^2b & 2a(2a^2+b^2) & 2a^2b & -4a^3 \\ ab^2 & -2a^2b & ab^2-2a^3 & 2a^2b & a(2a^2+b^2) & 0 \\ 0 & 0 & 0 & -4a^3 & 0 & 4a^3 \end{bmatrix} \tag{21}$$

Question 2. Adding all even rows results in:

$$\sum_{i=1}^{3} \boldsymbol{K}^e_{2i,j} = \frac{\pi E b}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

Odd rows add up to:

$$\sum_{i=1}^{3} \boldsymbol{K}^e_{2i+1,j} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Unlike in assignment 3, not both combinations equal to zero. This is due to symmetry. The vanishing of even rows means that forces on the Z axis must be balanced. On the r axis, however, forces need not be balanced. Since it is perpedicular to the axis of symmetry, any load in the radial direction compensates itself on the opposite side of Ω^3, even if it appears unbalanced in Ω^2.

Question 3. In order to compute the concentrated nodal forces we'll use its expression and simplify it:

$$\boldsymbol{f}^e = \int_{\Omega^3} \boldsymbol{N}(r,z)^T \boldsymbol{b} \, dV \tag{24}$$

$$= 2\pi \int_{\Omega^2} \boldsymbol{N}(r,z)^T \boldsymbol{b} r \, dS \tag{25}$$

$$= 2\pi \boldsymbol{N}(r_c, z_c)^T \boldsymbol{b} r_c \tag{26}$$

Where $\boldsymbol{N}$ is:

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{27}$$

This results in:

$$\boldsymbol{f}^e = -\frac{2\pi a^2 b g}{9} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \tag{28}$$

A Appendix

A.1 Matlab code

This code solves most of assignment 4.2.

```
%% Symbolic variables
a = sym('a','positive');
b = sym('b','positive');
E = sym('E','positive');
g = sym('g','real');

r = 2/3*a;
z = b/3;
S = a*b/2;

%% Symbolic matrices
B = 0*sym('B',[4,6]);
C = 0*sym('C',[4,4]);
N = 0*sym('N',[2,6]);
bf = 0*sym('bf',[2,1]); % Force vector

%% Filling matrices
bf = [0; -g];

N_1 = 1-r/a;
N_2 = r/a - z/b;
N_3 = z/b;

N = [N_1   0 N_2   0 N_3   0;
       0 N_1   0 N_2   0 N_3];

B = [       -1/a       0       1/a       0           0       0;
               0       0         0    -1/b           0     1/b;
           N_1/r       0     N_2/r       0       N_3/r       0;
               0    -1/a      -1/b     1/a         1/b       0];

C = E * [ 1 0 0 0;
          0 1 0 0;
          0 0 1 0;
          0 0 0 1/2];

%% Stiffness Matrix
K = B'*C*B * 2*pi*r * S;
K = simplify(K);

disp('K = ');
disp(K);
```

```
%% Force vector
f = N'*bf * 2*pi*r * S;

disp('f = ');
disp(f);

%% Stiffness Matrix's row sums
disp('Odd rows:')
sum = zeros(1,6);
for i=[1,3,5]
    sum = sum + K(i,:);
end
disp(simplify(sum))

disp('Even rows:');
sum = zeros(1,6);
for i=[2,4,6]
    sum = sum + K(i,:);
end
disp(simplify(sum))
```