Computational Structural Mechanics and Dynamics

<u>Assignment 4</u>

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MS in Computational Mechanics

Assignment 4.1

1. Compute the entries of K_e for the following axisymmetric triangle:

$$r_1 = 0,\ \ r_2 = r_3 = a,\ \ z_1 = z_2 = 0,\ \ z_3 = b$$

   The material is isotropic with $\nu = 0$ for which the stress-strain matrix is:

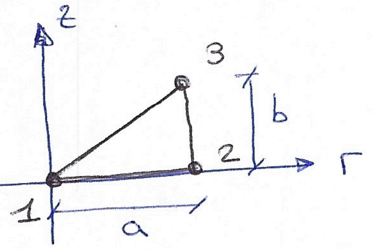
$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of K_e must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.
3. Compute the consistent force vector f_e for gravity forces $\boldsymbol{b} = [0, -g]^T$.

(Solutions attached in next pages)

1)

As we are working with a triangle element, the process to derive the stiffness equations of the 3-node triangle is going to be use, but taking into account the new system of coordinates



- It will be 2 dof per node, so that $u_i = \begin{Bmatrix} u_{ri}(r,z) \\ u_{zi}(r,z) \end{Bmatrix}$. As they are linear elements, the displacements can be expressed as

$$\left\{ \begin{aligned} u_{ri} &= a_1 + a_2 r_i + a_3 z_i \\ u_{zi} &= b_1 + b_2 r_i + b_3 z_i \end{aligned} \right\} \times 3$$

- Solving the unknowns, the displacement can be write as (for case u_{ri} as example)

$$\{u_{r}\} = [1 \; r \; z] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}[1 \; r \; z] \begin{bmatrix} (r_j z_k - z_j r_k) u_{ri} & (r_k z_i - z_k r_i) u_{rj} & (r_i z_j - z_i r_j) u_{rk} \\ (z_j - z_k) u_{ri} & u_{rj}(z_k - z_i) & (z_i - z_j) u_{rk} \\ (r_k - r_j) u_{ri} & u_{rj}(r_i - r_m) & u_{rk}(r_j - r_i) u_{rk} \end{bmatrix}$$

where the area $2A = (r_j z_k - r_k z_j) + (r_k z_i - r_i z_k) + (r_i z_j - r_j z_i)$

- As in the cartesian system, for the linear triangle, the displacements can be defined too as

$$\left\{ \begin{aligned} u_r &= u_{ri} N_i + u_{rj} N_j + u_{rk} N_k \\ u_z &= u_{zi} N_i + u_{zj} N_j + u_{rk} N_k \end{aligned} \right.$$

- Combining both definitions of u_r and u_z, the value of shape functions are optained.

- Writed for this case (index 1,2,3)

$$N_1 = \frac{1}{2A}\left[(r_2z_3 - z_2r_3) + z_{23}r + r_{32}z\right]$$

$$N_2 = \frac{1}{2A}\left[(r_3z_1 - r_1z_3) + z_{31}r + r_{13}z\right]$$

$$N_3 = \frac{1}{2A}\left[(r_1z_2 - z_1r_2) + z_{12}r + r_{21}z\right]$$

- Now, the different needed operators are defined.

$$D = \begin{bmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ 1/r & 0 \\ \partial/\partial z & \partial/\partial r \end{bmatrix} \qquad \underset{\substack{\downarrow \\ \text{Same for} \\ \text{nodes } 2,3}}{B_1} = DN = \begin{bmatrix} z_{23} & 0 \\ 0 & r_{32} \\ \frac{r_2z_3 - z_2r_3}{r} + z_{23} + \frac{r_{32}}{r}z & 0 \\ r_{32} & z_{23} \end{bmatrix}$$

- Finally as the constitutive matrix is know ([E]), the stiffness matrix can be defined and calculated

$$[k^e] = \iiint_V [B]^T[E][B]dV = 2\pi \iint_A [B]^T[D][B]\, r\,dr\,dz$$

Defining for this type of element
$$\begin{cases} r = \dfrac{r_1 + r_2 + r_3}{3} = \bar{r} \\ z = \dfrac{z_1 + z_2 + z_3}{3} = \bar{z} \\ A = \dfrac{b \cdot a}{2} \end{cases}$$

$$[k^e] = 2\pi \bar{r} A [\bar{B}]^T [E][\bar{B}]$$

- Substituting the values, and computing with Matlab help, the k^e matrix is obtained as:

$$[k^e] = \frac{2\pi E}{3b}\begin{bmatrix} 5b^2/4 & 0 & -3b^2/4 & 0 & b^2/4 & 0 \\ 0 & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ -3b^2/4 & ab/2 & a^2/2 + 5b^2/4 & -ab/2 & b^2/4 - a^2/2 & 0 \\ 0 & -b^2/2 & -ab/2 & a^2 + b^2/2 & (ab)/2 & -a^2 \\ b^2/4 & -ab/2 & b^2/4 - a^2/2 & ab/2 & a^2/2 + b^2/4 & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

b)

- Doing the sum of the columns and rows of the k^e matrix, writing the result in vector form (row and colum vectors respectively) the next results are obtained:

1. Sum of the columns:

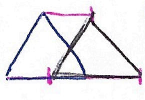
$$SC = \frac{2a^2 b\pi}{3}\begin{bmatrix} \frac{3b^2}{4} & 0 & \frac{3b^2}{4} & 0 & \frac{3b^2}{4} & 0 \end{bmatrix}$$

2. Sum of the rows:

$$SR = \frac{2a^2 b\pi}{3}\begin{bmatrix} \frac{3b^2}{4} & 0 & \frac{3b^2}{4} & 0 & \frac{3b^2}{4} & 0 \end{bmatrix}^T$$

- As it can be checked the sum of 2nd, 4th and 6th columns and rows are equal to zero, whereas the 1st, 3rd and 5th are not. Having a triangle element in a plane case, if the same displacement in each node for vertical and horizontal directions, the result will be that the element change his original position but it doesn't suffer any deformation:

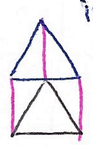
initial position



final position

displacement in x-direction

initial position



final position

displacement in y-direction

Whereas, working with an axisymmetric triangle, if the displacement is made in z-direction, any deformation is introduced, just a change on the element position, that means that any energy interchange has been made. But, if the displacement is in r-direction, due to the condition of axisymmetric, the discretized body will suffer a deformation in its shape with the corresponding energy dissipation. That's why the k^e possitions related to z-displacement summation is equal to zero, but not the relatives to r-displacement ones.

3)

- Defined the gravity forces $b = [0, -g]^T$

- For an axisymmetric element, working with cylindrical coordinates, body forces: $b(r,z) = \begin{bmatrix} b_r(r,z) \\ b_z(r,z) \end{bmatrix}$

- In this case, just the $b_z(r,z)$ component will be considered, so as: $b_z(r,z) = -g$

- To calculate the forces:

$$\{f_b\} = \iiint_V [N]^T \{b\} dV = 2\pi \iint_A [N]^T \{b\} r\, dA$$

- So for each node:

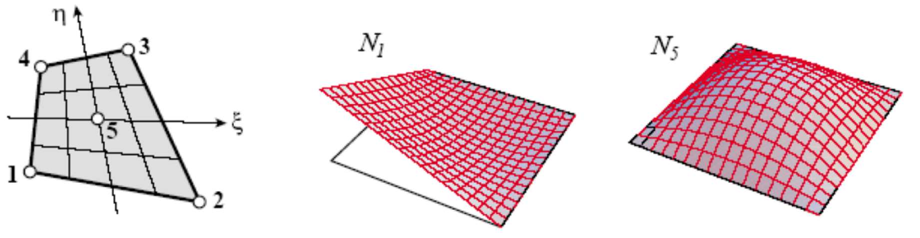
$$\{f_{bi}\} = \begin{bmatrix} f_{bir} \\ f_{biz} \end{bmatrix} = \frac{2\pi A \bar{r}}{3} \begin{bmatrix} 0 \\ b_{zi} \end{bmatrix}$$

- Finally the f^e vector, is computed:

$$\{f^e\} = \frac{2\pi a^2 b}{9} \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{bmatrix}$$

Assignment 4.2

A five node quadrilateral element has the nodal configuration shown in the figure. Perspective views of N_1^e and N_5^e are shown in the same figure.
Find five shape functions $N_i^e, i = 1, \ldots 5$ that satisfy compatibility and also verify that their sum is unity.



(Solutions attached in next pages

Isoparametric shape functions can be directly constructed by geometric considerations. The method is based on the observation that isoparametric functions are given as products of polynomial expressions in the natural coordinates, so that

$$N_i^e = C_i L_1 L_2 \dots L_m$$

where L_j are the homogeneous equations of lines or curves expressed as linear functions in natural coordinates. and C_i is the normalization coefficient, so that N_i^e has value 1 at the i^{th} node.

For 2D isoparametric elements, shape functions will be achieved following the established steps of the method.

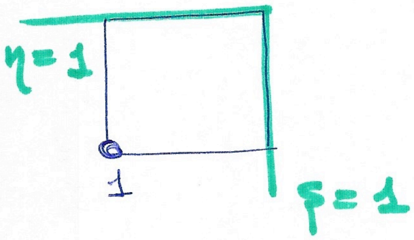
1) Select the L_j as the minimal number of lines that cross all the nodes expept the i^{th}.
2) Establish C_i; in order to $N_i^e = 1$ at i^{th}
3) N_i^e vanishes over all element sides that don't contain i.

- Looking the exercise case, a 5-node quadrilateral element, the steps to define the shape functions will be:
  1) Obtain N_i shape functions for $i = 1, 2, 3, 4$ as the 5th node doesn't exist.
  2) Obtain N_5 function.
  3) Combain the 4-node quadrilateral functions with N_5, in order to obtain the final result ($N_i = N_i + \alpha N_5$).

  The α parameter will be calculated to ensure that the final functions N_i meet their requirements.

1) SHAPE FUNCTIONS FOR 4-NODE QUADRILATERAL

- NODE 1



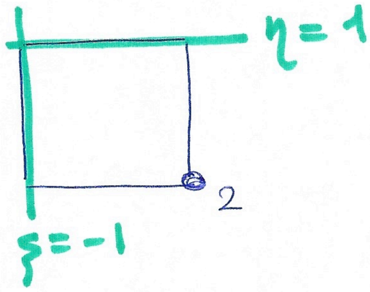
$$\left.\begin{array}{l} N_1^e = C_1 L_{23} L_{34} \\ L_{34}:\ \eta = 1 \\ L_{23}:\ \xi = 1 \end{array}\right\} \quad N_1^e = C_1(\xi - 1)(\eta - 1)$$

As $N(\xi, \eta) = N(-1, -1)$

$$1 = C_1(-1-1)(-1-1) = 4C_1 \qquad C_1 = \frac{1}{4}$$

$$\boxed{N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)}$$

- NODE 2



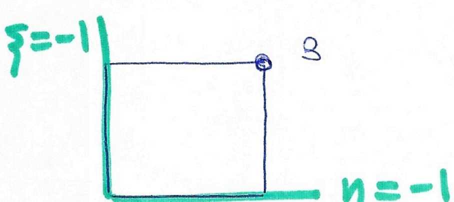
$$\left.\begin{array}{l} N_2^e = C_2 L_{34} L_{14} \\ L_{34}:\ \eta = 1 \\ L_{14}:\ \xi = -1 \end{array}\right\} \quad N_2 = C_2(\eta - 1)(\xi + 1)$$

As $N_2(\xi, \eta) = N_2(1, -1)$

$$1 = C_2(-1-1)(1+1) = -4C_2 \qquad C_2 = \frac{-1}{4}$$

$$\boxed{N_2^e = \frac{1}{4}(1 - \eta)(1 + \xi)}$$

- NODE 3



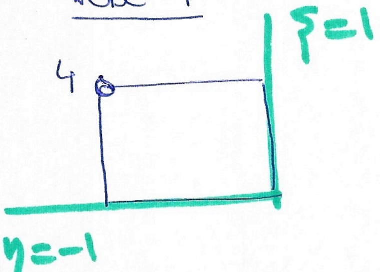
$$\left.\begin{array}{l} N_3^e = C_3 L_{14} L_{12} \\ L_{14}:\ \xi = -1 \\ L_{12}:\ \eta = -1 \end{array}\right\} \quad N_3 = C_3(\xi + 1)(\eta + 1)$$

As $N_3(\xi, \eta) = N_3(1, 1)$

$$1 = C_3(1+1)(1+1) = 4C_3 \qquad C_3 = \frac{1}{4}$$

$$\boxed{N_3^e = \frac{1}{4}(\xi + 1)(\eta + 1)}$$

• NODE 4



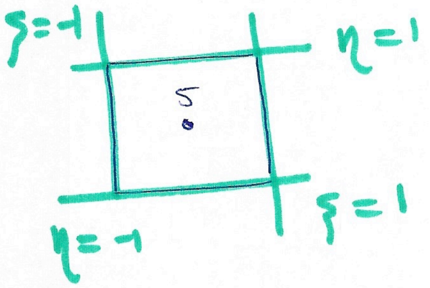
$$N_4^e = C_4 L_{23} L_{12}$$
$$L_{23}: \quad \xi = 1$$
$$L_{12}: \quad \eta = -1$$
$$\left.\right\} \quad N_4^e = C_4(\xi - 1)(\eta + 1)$$

As $N_4(\xi, \eta) = N_4(-1, 1)$

$$1 = C_4(-1-1)(1+1) = -4C_4 \qquad C_4 = \frac{-1}{4}$$

$$\boxed{N_4^e = \frac{1}{4}(1-\xi)(\eta+1)}$$

2) SHAPE FUNCTION N_5^e



$$N_5^e = C_5 L_{12} L_{23} L_{34} L_{14}$$
$$L_{12}: \quad \xi = -1$$
$$L_{23}: \quad \eta = 1$$
$$L_{34}: \quad \xi = 1$$
$$L_{14}: \quad \eta = -1$$

$$N_5^e = C_5(\xi+1)(\eta-1)(\xi-1)(\eta+1)$$

As $N_5(\xi, \eta) = N_5(0, 0)$

$$1 = C_5 \longrightarrow \quad N_5^e = (\xi+1)(\xi-1)(\eta-1)(\eta+1)$$

$$\boxed{N_5^e = (1-\eta^2)(1-\xi^2)}$$

3) COMBINED THE OBTAINED FUNCTIONS $\quad N_i^e = N_i^e + \alpha N_5^e$

- The value of α is going to be determined

* $N_1^e = \frac{1}{4}(\xi-1)(\eta-1) + \alpha(1-\eta^2)(1-\xi^2)$

At $N_5(0,0) \longrightarrow N_1^e = 0$

$$0 = \frac{1}{4}(0-1)(0-1) + \alpha(1-0)(1-0) \qquad \boxed{\alpha = \frac{-1}{4}}$$

$*$ $N_2^e = \frac{1}{4}(1-\eta)(1+\xi) + \alpha(1-\eta^2)(1-\xi^2)$

At node 5 $(0,0) \rightarrow N_2^e = 0$

$$0 = \frac{1}{4}(1-0)(1+0) + \alpha(1-0)(1-0) \qquad \boxed{\alpha = \frac{-1}{4}}$$

$*$ $N_3^e = \frac{1}{4}(\xi+1)(\eta+1) + \alpha(1-\xi^2)(1-\eta^2)$

At node 5 $(0,0) \rightarrow N_3^e = 0$

$$0 = \frac{1}{4}(0+1)(0+1) + \alpha(1-0)(1-0) \qquad \boxed{\alpha = \frac{-1}{4}}$$

$*$ $N_4^e = \frac{1}{4}(1-\xi)(1+\eta) + \alpha(1-\eta^2)(1-\xi^2)$

At node 5 $(0,0) \rightarrow N_4^e = 0$

$$0 = \frac{1}{4}(1-0)(1+0) + \alpha(1-0)(1-0) \qquad \boxed{\alpha = \frac{-1}{4}}$$

FINAL SHAPE FUNCTIONS :

$$N_1^e = \frac{1}{4}\left[(\xi-1)(\eta-1) - (1-\eta^2)(1-\xi^2)\right]$$

$$N_2^e = \frac{1}{4}\left[(1-\eta)(1+\xi) - (1-\eta^2)(1-\xi^2)\right]$$

$$N_3^e = \frac{1}{4}\left[(\xi+1)(\eta+1) - (1-\xi^2)(1-\eta^2)\right]$$

$$N_4^e = \frac{1}{4}\left[(1-\xi)(1+\eta) - (1-\eta^2)(1-\xi^2)\right]$$

$$N_5^e = (1-\eta^2)(1-\xi^2)$$

- SUM OF THE SHAPE FUNCTIONS.

- If local support and compatibility are satisfied, the summe of the obtained shape functions should be equal to one.

- The local support can be checked trying with any of the 5 shape functions the values achieved if any node coordinates are introduced. About the interelement compatibility, as we just have one element, the condition can be checked looking to the kind of shape functions achieved. As all of them are polynomial functions, they are going to be derivable.

- Now, the summ of the shape functions is done:

$$N_1^e + N_2^e + N_3^e + N_4^e + N_5^e =$$

$$\frac{1}{4}\left[(1-\eta)(1-\xi) + (1-\eta)(1+\xi) + (1+\eta)(1+\xi) + (1-\xi)(1+\eta)\right] +$$

$$+ \cancel{(1-\eta^2)}(1-\xi^2) - 4\frac{1}{4}\cancel{(1-\eta^2)}(1-\xi^2) =$$

$$= \frac{1}{4}(1-\xi-\eta+\eta\xi+1+\xi-\eta-\eta\xi+\xi\eta+\xi+\eta+1+1+\eta-\xi+\xi\eta) =$$

$$= \frac{4}{4} = 1$$

$$\boxed{N_1^e + N_2^e + N_3^e + N_4^e + N_5^e = 1}$$