Computational Structural Mechanics and Dynamics

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Assignment 4

Assignment 4.1

1]

$r_1 = 0$, $r_2 = r_3 = a$, $Z_1 = Z_2 = 0$, $Z_3 = b$

$\nu = 0$, stress-strain matrix =

$$E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Calculate stiffness matrix by Gauss Rule,

$$K^e = \int_{-1}^{1}\int_{-1}^{1} h\, B^T E B\, |J|\, d\xi\, d\eta$$

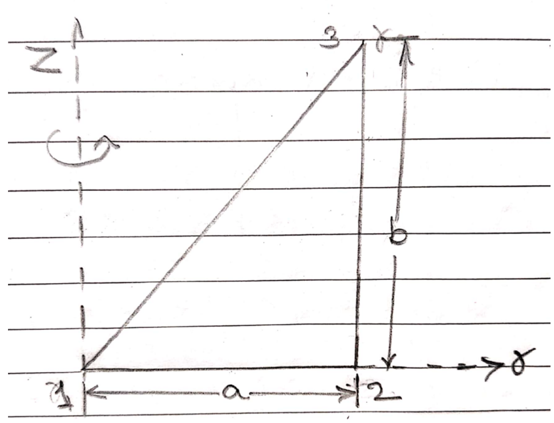
$$\approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j\, F(\xi, \eta)$$

For axis-symetric triangle

$$K^e = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\, B^T(\xi,\eta) E\, B(\xi_i,\eta_j)\, r(\xi_i,\eta_j)\, J(\xi_i,\eta_j) \quad \text{---} (1)$$

in which shape functions,

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1-\xi\eta \end{bmatrix}$$



$$B_i = \frac{1}{2A^e} \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix}$$

∴ *Shape function for axissymetric traingle*

$$N_i = \frac{1}{2A^e}\left[a_i + b_i r + C_i Z\right]$$

where,

$a_i = r_j Z_k - r_k Z_j$

$b_i = Z_j - Z_k$

$C_i = r_k - r_j$

Node	r	Z	a_i	b_i	C_i
1	0	0	ab	-b	0
2	a	0	0	b	-a
3	a	b	0	0	a

$\therefore \quad N_1 = \frac{1}{2A}(ab - br) = 1 - \frac{r}{a}$

$N_2 = \frac{1}{2A}(br - az) = \frac{r}{a} - \frac{z}{b}$ | $A = \frac{ab}{2}$

$N_3 = \frac{1}{2A}(az) = \frac{z}{b}$

$$B_i = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{a-r}{ar} & 0 & \frac{br-az}{abr} & 0 & \frac{z}{br} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

thus, the stiffness matrix becomes,

$$K^e = \int_A 2\pi E \begin{bmatrix} -\frac{2}{a} + \frac{2r}{a^2} + \frac{1}{r} & 0 & \frac{z}{ab} + \frac{1}{a} - \frac{2r}{a^2} - \frac{z}{br} & 0 & \frac{z}{br} - \frac{z}{ab} & 0 \\ 0 & \frac{r}{z^2} & \frac{r}{2ab} & -\frac{r}{2a^2} & -\frac{r}{2ab} & 0 \\ \frac{z}{ab} + \frac{1}{a} - \frac{2r}{a^2} - \frac{z}{br} & \frac{r}{2ab} & -\frac{2z}{ab} + \frac{2r}{a^2} + \frac{r}{2b^2} + \frac{z^2}{rb^2} & -\frac{r}{2ab} & \frac{z}{ab} - \frac{r}{2b^2} - \frac{z^2}{rb^2} & 0 \\ 0 & -\frac{r}{2a^2} & -\frac{r}{2ab} & \frac{r}{2a^2} + \frac{r}{b^2} & \frac{r}{2ab} & -\frac{r}{b^2} \\ \frac{z}{br} - \frac{z}{ab} & -\frac{r}{2ab} & \frac{z}{ab} - \frac{r}{2b^2} - \frac{z^2}{rb^2} & -\frac{r}{2ab} & \frac{r}{2b^2} + \frac{z^2}{rb^2} & 0 \\ 0 & 0 & 0 & -\frac{r}{b^2} & 0 & \frac{r}{b^2} \end{bmatrix} dA$$

Linear approximation,

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3$$
$$Z = N_1 Z_1 + N_2 Z_2 + N_3 Z_3$$

and natural coordinates,

$$r = (r_1 - r_3)\xi + (r_2 - r_3)\eta + r_3$$
$$z = (Z_1 - Z_3)\xi + (Z_2 - Z_3)\eta + Z_3$$

Deriving shape function with respect to ξ and η:

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial z} \end{bmatrix}$$

$$J = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

$$\therefore |J| = ab$$

The Gauss Quadrature selected was one point at the centroid which is $r = 2/3\,a$ and $z = b/3$ of the triangular element and $w_i = w_j = 1$

Substituting all in equation (1)

$$K^e = \frac{E}{2}\begin{bmatrix} \frac{5b}{3} & 0 & -b & 0 & \frac{b}{3} & 0 \\ 0 & \frac{2b}{3} & \frac{2a}{3} & -\frac{2b}{3} & -\frac{2a}{3} & 0 \\ -b & \frac{2a}{3} & \frac{2a^2}{3b}+\frac{5b}{3} & -\frac{2a}{3} & \frac{b}{3}-\frac{2a^2}{3b} & 0 \\ 0 & -\frac{2b}{3} & -\frac{2a}{3} & \frac{4a^2}{3b}+\frac{2b}{3} & \frac{2a}{3} & -\frac{4a^2}{3b} \\ \frac{b}{3} & -\frac{2a}{3} & \frac{b}{3}-\frac{2a^2}{3b} & \frac{2a}{3} & \frac{2a^2}{3b}+\frac{b}{3} & 0 \\ 0 & 0 & 0 & -\frac{4a^2}{3b} & 0 & \frac{4a^2}{3b} \end{bmatrix}$$

2]

Row and Column

2 : $\frac{2b}{3} + \frac{2b}{3} - \frac{2b}{3} - \frac{2a}{3} = 0$

4 : $-\frac{2b}{3} - \frac{2a}{3} + \frac{4a^2}{3b} + \frac{2b}{3} + \frac{2a}{3} - \frac{4a^2}{3b} = 0$

$6: \quad -\frac{4a^2}{3b} + \frac{4a^2}{3b} = 0$

thus it concludes that those nodes do not have constraints so it allow to have rigid body motion.

$1: \quad \frac{5b}{3} - b + \frac{b}{3} = b$

$3: \quad -b + \frac{2a}{3} + \frac{2a^2}{3b} + \frac{5b}{3} - \frac{2a}{3} + \frac{b}{3} - \frac{2a^2}{3b} = b$

$5: \quad \frac{b}{3} - \frac{2a}{3} + \frac{b}{3} - \frac{2a^2}{3b} + \frac{2a}{3} + \frac{2a^2}{3b} + \frac{b}{3} = b$

Due to restrictions in the matrix because of axis symmetric condition the nodes can only move in direction of r.

3]

Force vector for body forces

$$F_b = \int_A r N b \, dA$$

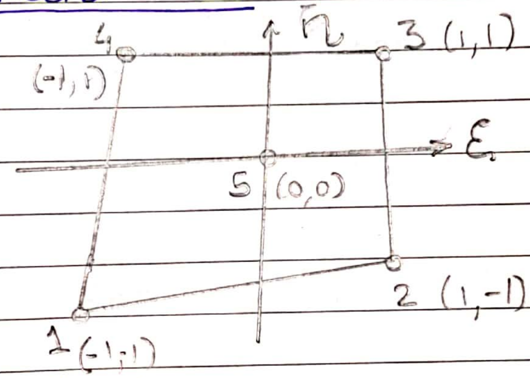
Substituting shape function and body force in z-direction

$$F_b = -\int_A g \begin{bmatrix} 0 \\ r - r^2/a \\ 0 \\ r^2/a - 2r/b \\ 0 \\ 2r/b \end{bmatrix} dA$$

$$f_b \simeq \sum_{i=1}^{P_1} w_i N_b |J|$$

$$\therefore \quad f_b = -g \begin{bmatrix} 0 \\ a^2b/9 \\ 0 \\ a^2b/9 \\ 0 \\ a^2b/9 \end{bmatrix}$$

Assignment 4.2



Shape functions

$N_i = N_i + a N_5$ where $i = 1, 2, 3, 4, 5$

where, $N_5 = \alpha L_{12} L_{2-3} L_{3-4} L_{4-1}$

$= \alpha (1+\eta)(1-\xi)(1-\eta)(1+\xi)$

$N_5 = a(1-\xi^2)(1-\eta^2)$

and for $a = 1$ at node 5,

$N_5 = 1$ & $N_1 = N_2 = N_3 = N_4 = 0$

$\therefore \quad N_1 = c(1-\xi)(1-\eta)(\xi)(\eta)$

$N_2 = c(1+\xi)(1-\eta)(\xi)(\eta)$

$N_3 = c(1+\xi)(1+\eta)(\xi)(\eta)$

$N_4 = c(1-\xi)(1+\eta)(\xi)(\eta)$

for $c = \frac{1}{4}$,

$N_1 = N_2 = N_3 = N_4 = 1$ & $N_5 = 0$