COMPUTATIONAL SUCTURAL MECHANICS AND DYNAMICS

Master of Science in Computational Mechanics/Numerical Methods

Spring Semester 2019

Rafel Perelló i Ribas

Assignment 4:

1. **Compute the entries of K^e for the following axisymmetric triangle:**

$$r_1 = 0, \qquad r_2 = r_3 = a, \qquad z_1 = z_2 = 0, \qquad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The vector of unknowns will be considered the following:

$$\boldsymbol{u} = \begin{bmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \\ u_{z1} \\ u_{z2} \\ u_{z3} \end{bmatrix}$$

In case of using a different vector of unknowns, the order of rows and columns of the stiffness matrix have to be permutated.

First, the basis functions are defined:

$$N_1^e = \xi, \qquad N_2^e = \eta, \qquad N_3^e = 1 - (\xi + \eta)$$

To compute the geometry and displacements unknowns the following linear relation is used:

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Now, the Jacobian is calculated:

$$\boldsymbol{J} = \frac{\partial(r,z)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \sum_{i=1}^{3} \begin{bmatrix} r_i \frac{\partial N_i^e}{\partial \xi} & z_i \frac{\partial N_i^e}{\partial \xi} \\ r_i \frac{\partial N_i^e}{\partial \eta} & z_i \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

$$\boldsymbol{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(r,z)} = \frac{1}{ab}\begin{bmatrix} -b & b \\ 0 & -a \end{bmatrix}$$

The strain vector is:

$$\boldsymbol{B} = \boldsymbol{DN} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1^e & N_2^e & N_3^e & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^e & N_2^e & N_3^e \end{bmatrix} = \begin{bmatrix} \boldsymbol{q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \boldsymbol{0} \\ \boldsymbol{q_z} & \boldsymbol{q_r} \end{bmatrix}$$

$$\boldsymbol{q_r} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial r} & \frac{\partial N_3^e}{\partial r} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial r} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

$$\boldsymbol{q_z} = \begin{bmatrix} \frac{\partial N_1^e}{\partial z} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_3^e}{\partial z} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial z} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -\frac{1}{b} & \frac{1}{b} \end{bmatrix}$$

$$\boldsymbol{q_\theta} = \begin{bmatrix} \frac{N_1^e}{r} & \frac{N_2^e}{r} & \frac{N_3^e}{r} \end{bmatrix} = \left(\sum_{i=1}^{3} r_i N_i^e \right)^{-1} \begin{bmatrix} N_1^e & N_2^e & N_3^e \end{bmatrix} =$$

$$= \frac{1}{a(1-\xi)} \begin{bmatrix} \xi & \eta & 1-(\xi+\eta) \end{bmatrix}$$

Finally, the elemental stiffness matrix computed as:

$$\boldsymbol{K}^e = \int_0^1 \int_0^{1-\xi} \boldsymbol{G}(\xi,\eta) d\eta d\xi$$

Where:

$$\boldsymbol{G}(\xi,\eta) = \boldsymbol{B}^T(\xi,\eta)\boldsymbol{E}\boldsymbol{B}(\xi,\eta)r(\xi,\eta)\boldsymbol{J}(\xi,\eta) =$$

$$= \begin{bmatrix} \boldsymbol{q_r^T} & \boldsymbol{0} & \boldsymbol{q_\theta^T} & \boldsymbol{q_z^T} \\ \boldsymbol{0} & \boldsymbol{q_z^T} & \boldsymbol{0} & \boldsymbol{q_r^T} \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \boldsymbol{0} \\ \boldsymbol{q_z} & \boldsymbol{q_r} \end{bmatrix} \left(\sum_{i=1}^{3} r_i N_i^e \right) \left\| \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \right\| =$$

$$= E \begin{bmatrix} \boldsymbol{q_r^T} & \boldsymbol{0} & \boldsymbol{q_\theta^T} & \boldsymbol{q_z^T} \\ \boldsymbol{0} & \boldsymbol{q_z^T} & \boldsymbol{0} & \boldsymbol{q_r^T} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \boldsymbol{0} \\ \frac{1}{2}\boldsymbol{q_z} & \frac{1}{2}\boldsymbol{q_r} \end{bmatrix} a(1-\xi)ab =$$

$$= E\, a^2 b(1-\xi) \begin{bmatrix} \boldsymbol{q_r^T q_r} + \boldsymbol{q_\theta^T q_\theta} + \frac{1}{2}\boldsymbol{q_z^T q_z} & \frac{1}{2}\boldsymbol{q_z^T q_r} \\ \frac{1}{2}\boldsymbol{q_r^T q_z} & \boldsymbol{q_z^T q_z} + \frac{1}{2}\boldsymbol{q_r^T q_r} \end{bmatrix}$$

The integral of this term is computed using numerical integration. In plane state, the degree of the quadrature needed for this integration would be 2. However, due to the radius term multiplying the matrix, in revolution linear elements, a degree 3 quadrature is needed. In this particular case, as the vectors $\boldsymbol{q_r}$ and $\boldsymbol{q_z}$ are constant and the radius term multiplying simplifies with the radius term dividing in $\boldsymbol{q_\theta}$, a 2 degree quadrature could be used and as the radius is dividing in some terms an open quadrature ha to be chosen (Gauss for example). In any case, as a rational term is present in $\boldsymbol{q_\theta}$, the quadrature will not yield to exact values.

2. **Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K}^{\mathrm{e}}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 do not vanish, and explain why.**

   In this question it has been assumed that the vector of unknowns is:

$$\boldsymbol{u} = \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix}$$

This differs from the way the stiffness matrix was calculated in the previous point. For that reason, the sum of rows (and columns) 4, 5 and 6th of $\mathbf{K^e}$ do vanish while the rest not.

This is easily seen noting that the sum of the vectors $\boldsymbol{q_r}$ and $\boldsymbol{q_z}$ vanish. This is due to the definition of the basis functions: their sum is 1 over all the points over the element. That is why the sum of their derivatives is 0.

As the 4, 5 and 6 rows of the elemental stiffness matrix are composed only of tensorial products of this two vectors their sum is null.

The 1, 2 and 3rd rows contain terms of tensorial product of $\boldsymbol{q_\theta}$ this vector is composed of the shape functions divided by the radius. As both terms are positive over all the element, their sum will be positive over the whole element so it cannot vanish after the integration.

As the matrix is symmetric, this same reasoning applies to the columns.

The physical reason of that phenomenon is that the sum of all the columns represent the nodal force at each degree of freedom for unitary nodal displacements in all the degrees of freedom. In the z-coordinate this is translated in a simply rigid body translation so it does not produce any reaction. However, in the case of the r-coordinate it is translated to a straining of the material in the azimuthal direction (represented in the $\boldsymbol{q_\theta}$ term) as the radius of circumference would increase. That is why there is a reaction against this action.

3. **Compute the consistent force vector $\mathbf{f}^{\mathbf{e}}$ for gravity forces $\mathbf{b} = [\mathbf{0}, -\mathbf{g}]^{\mathbf{T}}$**

The consistent force vector is computed as:

$$\boldsymbol{f}^e = \int_0^1 \int_0^{1-\xi} \boldsymbol{N}^{\boldsymbol{T}} \boldsymbol{b} r J d\eta d\xi =$$

$$= \int_0^1 \int_0^{1-\xi} \begin{bmatrix} \xi & 0 \\ \eta & 0 \\ 1-(\xi+\eta) & 0 \\ 0 & \xi \\ 0 & \eta \\ 0 & 1-(\xi+\eta) \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} a(1-\xi) \left| \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \right| d\eta d\xi =$$

$$= a^2 bg \int_0^1 \int_0^{1-\xi} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi \\ -\eta \\ -1+(\xi+\eta) \end{bmatrix} (1-\xi) d\eta d\xi =$$

$$= a^2 bg \int_0^1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi \cdot (1-\xi)^2 \\ -\dfrac{(1-\xi)^3}{2} \\ -(1-\xi)^2 + \dfrac{(1-\xi)^3}{2} \end{bmatrix} d\xi =$$

$$\boldsymbol{f}^e = a^2 bg \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{1}{12} \\ -\dfrac{1}{8} \\ -\dfrac{5}{24} \end{bmatrix}$$

4. **A five node quadrilateral element has the nodal configuration shown in the figure. Perspective views of $\mathbf{N_1^e}$ and $\mathbf{N_5^e}$ are shown in the figure.**
   Find five shape functions $\mathbf{N_i^e}, \boldsymbol{i} = \mathbf{1}, \ldots, \mathbf{5}$ that satisfy compatibility and also verify that their sum is unity.

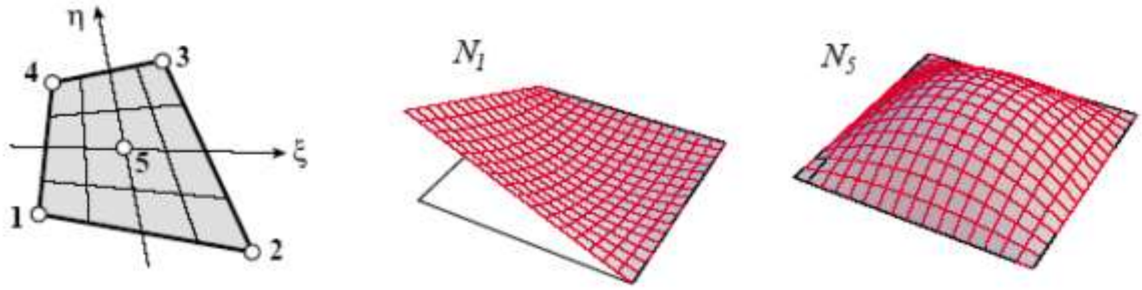

Figure.- Five node quadrilateral element

First, a shape function that vanishes at the boundaries is desired for the new degree of freedom. This bubble function is assumed to be biquadratic. Legendre interpolations are used:

$$N_5^e = I_1(\xi) \cdot I_2(\eta) = -\big((\xi+1)\cdot(\xi-1)\big) \cdot -\big((\eta+1)\cdot(\eta-1)\big) =$$
$$= (1+\xi)(1-\xi)(1+\eta)(1-\eta)$$

As this basis function vanishes at the boundaries of the elements it can be added to the original 4 basis functions without affecting the compatibility of the displacements. So in order that the sum of the five is the unity, the new basis function will be subtracted to the original ones equally:

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1+\xi)(1-\xi)(1+\eta)(1-\eta)$$
$$= \frac{1}{4}(1-\xi)(1-\eta)\Big(1 - \big((1+\xi)(1+\eta)\big)\Big)$$
$$N_2^e = \frac{1}{4}(1+\xi)(1-\eta)\Big(1 - \big((1-\xi)(1+\eta)\big)\Big)$$
$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta)\Big(1 - \big((1-\xi)(1-\eta)\big)\Big)$$
$$N_4^e = \frac{1}{4}(1-\xi)(1+\eta)\Big(1 - \big((1+\xi)(1-\eta)\big)\Big)$$
$$N_5^e = (1+\xi)(1-\xi)(1+\eta)(1-\eta)$$

$$N_1^e + N_2^e + N_3^e + N_4^e + N_5^e = \frac{1}{4}((1-\xi)(1-\eta) + (1+\xi)(1-\eta) +$$
$$+(1+\xi)(1+\eta) + (1-\xi)(1+\eta)) - (1+\xi)(1-\xi)(1+\eta)(1-\eta)$$
$$+ (1+\xi)(1-\xi)(1+\eta)(1-\eta) =$$
$$\frac{1}{4}\big((1+\xi)(1+\eta+1-\eta) + (1-\xi)(1+\eta+1-\eta)\big) = \frac{1}{4}\big(2(1+\xi+1-\xi)\big) = 1$$