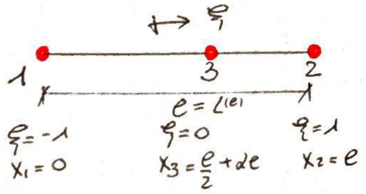
ASSIGNMENT 4

ASSIGNMENT 4.1.

3-node straight bar element. length $\ell = x_1 - x_2$. Isoparametric definition:

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \qquad N_i^e(\xi) = \text{shape functions.}$$

$$\begin{cases} x_1 = 0 \\ x_2 = \ell \\ x_3 = \left(\frac{1}{2} + \alpha\right)\ell \qquad -\frac{1}{2} < \alpha < \frac{1}{2} \end{cases}$$



(i) Get Jacobian $J = \frac{dx}{d\xi}$. Show that: $\begin{cases} - \text{ if } -\frac{1}{4} < \alpha < \frac{1}{4} \text{ and } J > 0 \text{ over the whole element } -1 \leq \xi \leq 1. \\ - \text{ if } \alpha = 0,\ J = \frac{1}{2} \text{ is a constant over the element.} \end{cases}$

We obtain the shape functions by using Lagrange polynomials.

- $N_1 = \frac{(\xi - 0)(\xi - 1)}{(-1)(-1-1)} = \frac{1}{2}\xi(\xi - 1)$
- $N_2 = \frac{(\xi+1)(\xi+0)}{(1+1)(1+0)} = \frac{1}{2}\xi(\xi+1) \quad \Rightarrow \quad x = \sum N_i x_i = N_1x_1 + N_2x_2 + N_3x_3$
- $N_3 = \frac{(\xi+1)(\xi-1)}{(1)(-1)} = 1 - \xi^2$

$$= \frac{\xi e}{2}(\xi+1) + (1-\xi^2)\left[\frac{e}{2} + \alpha e\right]$$

So the Jacobian will be:

$$\frac{dx}{d\xi} = \frac{e}{2}(2\xi + 1) - 2\xi\left(\frac{e}{2} + \alpha e\right)$$

- if $-\frac{1}{4} < \alpha < \frac{1}{4}$ and $J > 0 \longrightarrow \boxed{\alpha = -\frac{1}{4}}$ for whole element $J > 0$ as show.

$$J = \frac{1}{2}(2\xi+1) - 2\xi\left(\frac{\ell}{2}-\frac{\ell}{4}\right) = \frac{\ell}{2}\left[\cancel{2\xi}+1-\cancel{2\xi}\right] = \frac{\ell}{2}$$

So:

$$J = \frac{\ell}{2} - \frac{2\xi\alpha}{\ell} > 0$$

– if $\alpha = 0 \longrightarrow J = \frac{\ell}{2}$ constant!

(ii) Obtain the 1×3 strain displacement matrix $\mathbb{B}$ relating $e = \frac{du}{dx} = \mathbb{B}u^e$ where u^e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of $\mathbb{B}$ are functions of $1, \alpha, \xi$.

Hint: $\mathbb{B} = \frac{d\mathbb{N}}{dx} = J^{-1}\frac{d\mathbb{N}}{d\xi}$ where $\begin{cases} \mathbb{N} = [N_1\ N_2\ N_3] \\ J = \text{Jacobian} \end{cases}$

$$\mathbb{B} = \frac{1}{\ell\left(\frac{1}{2} - 2\xi\alpha\right)}\left[\frac{1}{2}(2\xi-1) \quad \frac{1}{2}(2\xi+1) \quad -2\xi\right]$$

$$= J^{-1}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi} \quad \frac{dN_3}{d\xi}\right].$$

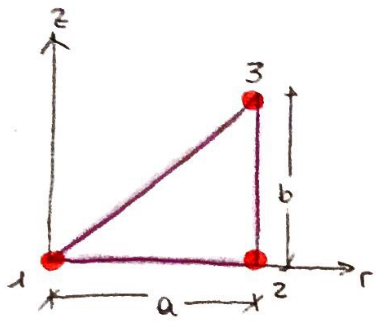
ASSIGNMENT 4.2

(a) Compute the entries of $\mathbb{K}^e$ for the following axisymmetric triangle:

$r_1 = 0 \qquad r_2 = r_3 = a \qquad z_1 = z_2 = 0 \qquad z_3 = b$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is:

$$\mathbb{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$



By the isoparametric formulation we have.

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix}$$

ξ, η = natural coordinates.

We need a coordinate transformation because the shape functions are given on natural coordinates.

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} \partial N_i/\partial r \\ \partial N_i/\partial z \end{bmatrix} = J^{-1} \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix} \qquad \text{where } J = \text{Jacobian} = \begin{bmatrix} \partial r/\partial \xi & \partial z/\partial \xi \\ \partial r/\partial \eta & \partial z/\partial \eta \end{bmatrix}$$

$$J = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

- Calculate the derivative of the shape functions: (natural coordinates)

$$\begin{bmatrix} \partial N_1/\partial \xi \\ \partial N_1/\partial \eta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \partial N_2/\partial \xi \\ \partial N_2/\partial \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \partial N_3/\partial \xi \\ \partial N_3/\partial \eta \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

so the vector N is defined below:

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Global coordinates:

$$\begin{cases} N_1 = 1 - \dfrac{r}{a} \\ N_2 = \dfrac{r}{a} - \dfrac{z}{b} \\ N_3 = \dfrac{z}{b} \end{cases}$$

. We know that the matrix $\mathbb{B}$ is defined by:

$$\mathbb{B} = \mathbb{D} \times \mathbb{N}$$

And so:

$$\mathbb{B} = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ 1/r - 1/a & 0 & 1/a - z/rb & 0 & z/rb & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

And the STIFFNESS-MATRIX $\mathbb{K}^e = \int_{\Omega} 2\pi \, r \, \mathbb{B}^T \mathbb{E} \mathbb{B} \, dA$

for the axisymmetry we can reduce the volume integral of the 3D triangle into an area integral x the circumference.

$$\downarrow$$

$$2\pi \int_{\Omega} r \, (r/z) \, dA$$

So we have:

Jacobian matrix

$$\mathbb{K}^e = 2\pi \int_{\Omega} r \, \mathbb{B}^T \mathbb{E} \mathbb{B} \, dA = 2\pi \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} r \, \mathbb{B}^T \mathbb{E} \mathbb{B} \det J \, d\xi d\eta$$

natural coordinate

To ~~calculate~~ integrate the stiffness matrix we use the one-point Gauss integration in which the evaluated function must be ~~ev~~ integrated on the middle of the natural domain (zero).

$$\mathbb{K}^e = \mathbb{E} \begin{bmatrix} 5b/12 & 0 & -b/4 & 0 & b/12 & 0 \\ 0 & b/6 & a/6 & -b/6 & -a/6 & 0 \\ -b/4 & a/6 & 5b/12 + a^2/6b & -a/6 & b/12 - a^2/6b & 0 \\ 0 & -b/6 & -a/6 & b/6 + a^2/3b & a/6 & -a^2/3b \\ b/12 & -a/6 & b/12 - a^2/6b & a/6 & b/12 + a^2/6b & 0 \\ 0 & 0 & 0 & -a^2/3b & 0 & a^2/3b \end{bmatrix}$$

(2) Show that the sum of the rows (/ columns) 2, 4, 6 of K^e must be vanish and explain why.
Show as well that the sum of rows (/ columns) 1, 3, 5 does not vanish, and explain why.

Sum of roms 2, 4, 6:

$$[b/6 + a/6 + (-b/6) + (-a/6)] + \quad \text{row 2}$$

$$+ [-b/6 - a/6 + b/6 + a^2/3b + a/6 - a^2/3b] + \quad \text{row 4}$$

$$+ [-a^2/3b + a^2/3b] = \quad \text{row 6}$$

$= 0\,!$ ← This is because the element could have a rigid motion inside the model.

The same for the columns 2, 4, 6.

Sum of rows 1, 3, 5:

$$[5b/12 + (-b/4) + b/12] +$$

$$+ [-b/4 + a/6 + [5b/12 + a^2/6b] - a/6 + b/12 - a^2/6b] +$$

$$+ [b/12 - a/6 + b/12 - a^2/6b + a/6 + b/12 + a^2/6b] \neq 0.$$

This because the equations inside rows 1, 3, 5 ensure that the symmetry will be not violated, which would occur if there is a movement of the rigid body.

(3) Compute the consistent force vector f^e for gravity forces $b = [0, -g]^T$.

As the stiffness matrix, we can calculate the consistent force vector by using one-point Gauss integration.

$$f = \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{p} W_k W_l N_{k,l}^T \, b_{k,l} \det J_{k,l}$$

and we obtain:

$$f = \begin{bmatrix} 0 \\ -a^2 bg/9 \\ 0 \\ -a^2 bg/9 \\ 0 \\ -a^2 bg/9 \end{bmatrix}$$