# Assignment 5

PRADEEP KUMAR BAL

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### Problem 5.1

We are given a three-nodded bar element which is referred to the natural coordinate $\xi$. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1, \xi = 1 and\ \xi = 0$, respectively. The variation of the shape functions $N_1(\xi),\ N_2(\xi)\ and\ N_3(\xi)$ is sketched in the figure below. The given shape functions are the quadratic polynomials in $\xi$: $N_1{}^{\mathbf{e}}(\xi) = a_0 + a_1\ \xi + a_2\ \xi^2, N_2{}^{\mathbf{e}}(\xi) = b_0 + b_1\ \xi + b_2\ \xi^2 and N_3{}^{\mathbf{e}}(\xi) = c_0 + c_1\ \xi + c_2\ \xi^2$.

(a) For the shape function $N_1{}^{\mathbf{e}}(\xi)$, using the node value conditions,
$N_1{}^{\mathbf{e}}(\xi = -1) = 1, N_1{}^{\mathbf{e}}(\xi = 0) = 0\ and\ N_1{}^{\mathbf{e}}(\xi = 1) = 0.$

So we can write
At $\xi = -1$;
$a_0 - a_1 + a_2 = 1;$
$At\ \xi = 0;$
$a_0 = 0;$
$At\ \xi = 1;$
$a_0 + a_1 + a_2 = 0$

On solving the above three equations:
$a_0 = 0;\ a_1 = -1/2;\ a_2 = 1/2;$
$N_1{}^e(\xi) = \frac{\xi\ (\xi-1)}{2}$

Similarly; for $N_2{}^e(\xi),\ \ N_2{}^e(\xi = -1) = 0;\ N_2{}^e(\xi = 0) = 0; N_2{}^e(\xi = 1) = 1;$

We can write:
$b_0\ -\ b_1\ +\ b_2 = 0$
$b_0 = 0;$
$b_0\ +\ b_1\ +\ b_2 = 1$
On solving these three above equations we get
$b_0 = 0; b_1\ = b_2\ = 1/2;$

$N_2{}^e(\xi) = \frac{\xi\ (\xi+1)}{2}$

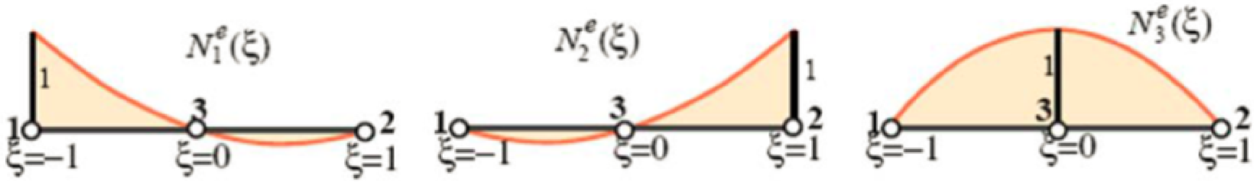



Figure 1: Isoparametric shape functions for 3-node bar element

For ${N_3}^e(\xi)$,
${N_3}^e(\xi = -1) = 0; {N_3}^e(\xi = 0) = 1 \ {N_3}^e(\xi = 1) = 0$

So, we can write
$c_0 - c_1 + c_2 = 0$
$c_0 = 1;$
$c_0 + c_1 + c_2 \ = \ 0$

On solving the above three equations we get $c_0 = 1, \ c_1 = 0; c_2 = -1;$
${N_3}^e(\xi) = 1 - \xi^2$

(b) ${N_1}^e(\xi) + {N_2}^e(\xi) \ + \ {N_1}^e(\xi) \ = \ \frac{\xi \ (\xi-1)}{2} + \ \frac{\xi \ (\xi+1)}{2} + \ 1 - \xi^2{=}1;$

It verifies that the sum of the three shape functions is 1.

The derivatives of the shape functions with respect to the natural co-ordinates are:

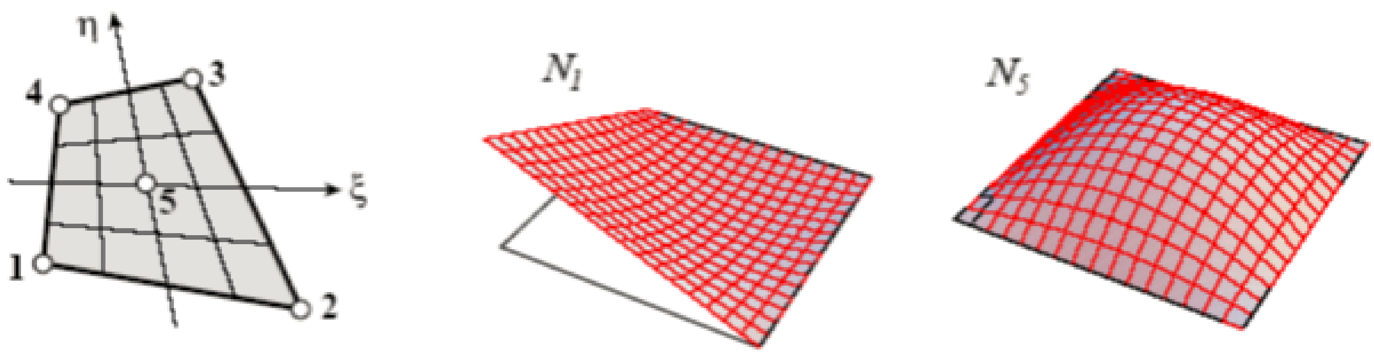
(c) $\frac{d \ {N_1}^e}{d\xi} \ = \ \frac{d \ (\frac{\xi \ (\xi-1)}{2})}{d\xi} \ = \ \xi - 1/2$

$\frac{d \ {N_2}^e}{d\xi} \ = \ \frac{d \ (\frac{\xi \ (\xi+1)}{2})}{d\xi} \ = \ \xi + 1/2$

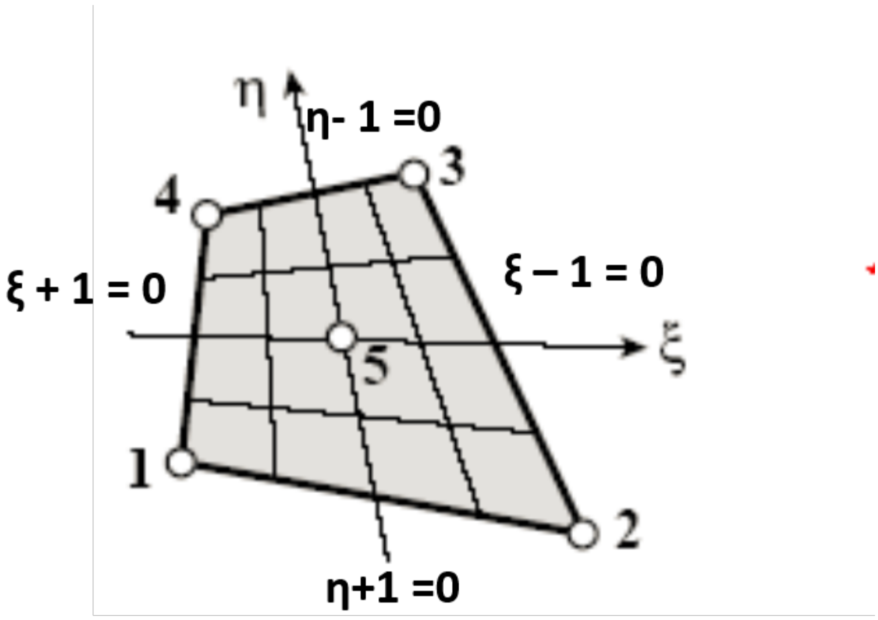
$\frac{d \ {N_3}^e}{d\xi} \ = \ \frac{d \ (1-\xi^2)}{d\xi} \ = \ -2 \ \xi$

**Problem 5.2**

A five node quadrilateral element has the nodal configuration shown in the figure below with two perspective views of ${N_1}^e$ *and* ${N_5}^e$.




In the natural co-ordinates the sides can be represented as shown in the figure below:




Using the line product method $N_5(\xi,\eta)$ can be represented as:

$N_5(\xi,\eta) = c_5\ L_{1-2}\ L_{2-3}\ L_{3-4}\ L_{4-1};$

$So, N_5(\xi,\eta) = c_5\ (\eta+1)(\xi+1)(\eta-1)(\xi-1)$

But $N_5(\xi=0,\eta=0) = 1$

It gives us $c_5 = 1$; So,

$N_5(\xi,\eta) = (\eta+1)(\xi+1)(\eta-1)(\xi-1) = (\xi^2-1)\ (\eta^2-1)$

We know that the corner shape functions for a four nodded quadrilateral element are:

$\underline{N_1} = \frac{(1-\xi)(1-\eta)}{4}$

$\underline{N_2} = \frac{(1+\xi)(1-\eta)}{4}$

$\underline{N_3} = \frac{(1+\xi)(1+\eta)}{4}$

$\underline{N_4} = \frac{(1-\xi)(1+\eta)}{4}$

For $i = 1,2,3,4$ for the given 5 nodded quadrilateral element it is assumed that :

$N_i = \underline{N_i} + \alpha\ N_5\ \ and\ N_i(\xi=0,\eta=0) = 0\ at\ the\ node\ 5$

$For\ i = 1;\ N_1 = \underline{N_1} + \alpha\ N_5$

$N_1 = \frac{(1-\xi)(1-\eta)}{4} + \alpha(\eta^2-1)\ (\xi^2-1);$

$Implementing\ N_1(\xi=0,\eta=0) = 0; we\ have\ \alpha = -1/4$

*So,*

$N_1 = \frac{(1-\xi)(1-\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4};$

$N_1(\xi,\eta) = -\frac{(1-\xi)(1-\eta)\ (\xi+\eta+\xi\eta)}{4}$

$For\ i = 2;\ N_2 = \underline{N_2} + \alpha\ N_5$

$N_2 = \frac{(1+\xi)(1-\eta)}{4} + \alpha(\eta^2-1)\ (\xi^2-1);$

$Implementing\ N_2(\xi = 0, \eta = 0) = 0; we\ have\ \alpha = -1/4$

$So,$

$N_2 = \frac{(1+\xi)(1-\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4};$

$N_2(\xi,\eta) = \frac{(1+\xi)(1-\eta)\ (-\eta+\xi+\xi\eta)}{4}$

**For** $i = 3;\ N_3 = \underline{N_3} + \alpha\ N_5$

$N_3 = \frac{(1+\xi)(1+\eta)}{4} + \alpha(\eta^2-1)\ (\xi^2-1);$

$Implementing\ N_3(\xi = 0, \eta = 0) = 0; wehave\alpha = -1/4$

$So,$
$N_3 = \frac{(1+\xi)(1+\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4};$

$N_3(\xi,\eta) = \frac{(1+\xi)(1+\eta)\ (\xi+\eta-\xi\eta)}{4}$

**For** $i = 4;\ \ N_4 = \underline{N_4} + \alpha\ N_5$

$N_4 = \frac{(1-\xi)(1+\eta)}{4} + \alpha(\eta^2-1)\ (\xi^2-1);$

$Implementing\ N_4(\xi = 0,\ \eta = 0) = 0; we\ have\ \alpha = -1/4$

$So,$
$N_4 = \frac{(1-\xi)(1+\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4};$

$N_4(\xi,\eta) = \frac{(1-\xi)(1+\eta)\ (\eta-\xi+\xi\eta)}{4}$

$N_1(\xi,\eta) + N_2(\xi,\eta) + N_3(\xi,\eta) + N_4(\xi,\eta) + N_5(\xi,\eta)$

$= \frac{(1-\xi)(1-\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4} + \frac{(1+\xi)(1-\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4} + \frac{(1+\xi)(1+\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4} + \frac{(1-\xi)(1+\eta)}{4} - \frac{(\eta^2-1)\ (\xi^2-1)}{4} + (\eta^2-1)\ (\xi^2-1) = 1$

So, it verifies that$N_1 + N_2 + N_3 + N_4 + N_5 = 1$

**Problem 5.3**

1. The 8-node hexahedron:

The $2 * 2 * 2$ *rules since*, $2^3 * 6 = 48 > 8 * 3 - 6 = 18$ gives a rank sufficient stiffness matrix.

2. The 20-node hexahedron:

The $3 * 3 * 3$ *rules since*, $3^3 * 6 = 162 > 20 * 3 - 6 = 54$ gives a rank sufficient stiffness matrix.

3. The 27-node hexahedron:

The $3 * 3 * 3$ *rules since*, $3^3 * 6 = 162 > 27 * 3 - 6 = 75$ gives a rank sufficient stiffness matrix.

4. The 64-node hexahedron:

The $4 * 4 * 4$ *rules since*, $4^3 * 6 = 384 > 64 * 3 - 6 = 186$ gives a rank sufficient stiffness matrix.