Computational Structural Mech. & Dynamics Marcos Baquer

Isoparametric Representation ASSIGNMENT 5

Assignment 5.1

3 node bar referred to the natural coordinate ξ, while the two end nodes and the mid node are 1, 2, 3 respectively.

1) $\xi = -1$
2) $\xi = 1$
3) $\xi = 0$

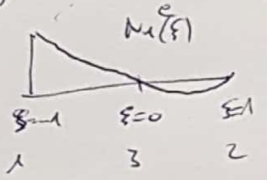
The functions are quadratic polynomials.

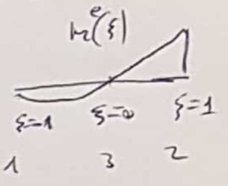
$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$

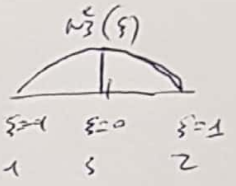
$N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2$

$N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$

a) Determine $a_0, \dots c_2$







$$\begin{cases} 1 = a_0 - a_1 + a_2 \\ 0 = a_0 \\ 0 = 0 + a_1 + a_2 \end{cases}$$

$N_1^e(\xi) = 1$ when $\xi = -1$
$N_1^e(\xi) = 0$ when $\xi = 0$
$N_1^e(\xi) = 0$ when $\xi = 1$

$\downarrow$

$a_0 = 0$
$a_1 = -1/2$
$a_2 = 1/2$

$\boxed{N_1^e(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2} = \frac{1}{2}(\xi^2 - \xi) = \boxed{\frac{1}{2}\xi(\xi-1)}$

$$\begin{cases} 0 = b_0 - b_1 + b_2 \\ 0 = b_0 + b_1 \\ 1 = b_0 + b_1 + b_2 \end{cases}$$

$N_2^e(\xi) = 0$ when $\xi = -1$
$N_2^e(\xi) = 0$ when $\xi = 0$
$N_2^e(\xi) = 1$ when $\xi = 1$

$\rightarrow \boxed{b_0 = 0 \quad b_1 = \frac{1}{2} \quad b_2 = \frac{1}{2}}$

$\boxed{N_2^e(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2} = \boxed{\frac{1}{2}\xi(\xi+1)}$

$$\begin{cases} 0 = c_0 - c_1 + c_2 \\ 1 = c_0 \\ 0 = c_0 + c_1 + c_2 \end{cases}$$

$N_3^e(\xi) = 0$ when $\xi = -1$
$N_3^e(\xi) = 1$ when $\xi = 0$
$N_3^e(\xi) = 0$ when $\xi = 1$

$\rightarrow$

$c_0 = 1$
$c_1 = 0$
$c_2 = -1$

$\boxed{N_3^e(\xi) = \boxed{1 - \xi^2}}$

b) $-\cancel{\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + \cancel{\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + 1 - \cancel{\xi^2} = \boxed{1}$

c) Derivatives respect to the natural coordinates:

$$\boxed{\begin{aligned} \frac{\partial N_1}{\partial \xi} &= -\frac{1}{2} + \xi \\ \frac{\partial N_2}{\partial \xi} &= \frac{1}{2} + \xi \\ \frac{\partial N_3}{\partial \xi} &= 1 - 2\xi \end{aligned}}$$

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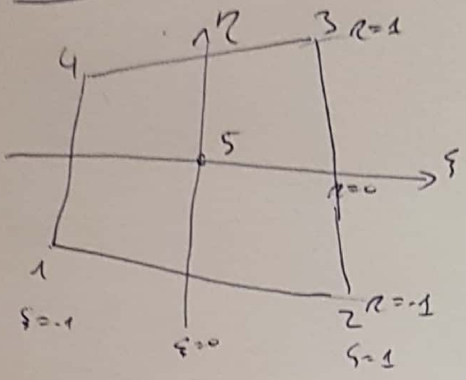
Problem 5.2



	ξ_i	η_i
1	-1	-1
2	+1	-1
3	+1	+1
4	-1	+1

linear $\begin{cases} N_1 = \frac{1}{2}(1-\xi) & \xi=-1 \\ N_2 = \frac{1}{2}(1+\xi) & \xi = 1 \end{cases}$

$$N_1(\xi,\eta) = N_1^e(\xi) N_1^e(\eta) = \frac{1}{2}(1-\xi)\cdot\frac{1}{2}(1-\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$\xi=-1$ linear
$\eta=-1$ linear

$$N_2^e(\xi,\eta) = N_2^e(\xi) N_2^e(\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1-\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$\xi=1$ linear
$\eta=-1$

$$N_3^e(\xi,\eta) = N_3^e(\xi) N_3^e(\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1+\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$\xi=1$
$\eta=1$ linear

$$N_4^e(\xi,\eta) = N_4^e(\xi) N_4^e(\eta) = \frac{1}{2}(1-\xi)\frac{1}{2}(1+\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

$\xi=1$
$\eta=1$ linear

$$\underset{i=1\ldots4}{N_i^e}(\xi,\eta) = \frac{1}{4}(1-\xi_i)(1-\eta_i)$$

$$N_5^e(\xi\eta) = N_5^e(\xi) N_5^e(\eta) = \boxed{\boxed{(1-\xi^2)(1-\eta^2)}}$$

in corners it's directly 0.

$\xi=0$
$\eta=0$
quadratic

quadratic $N_1 = 1-\xi^2$ $\xi=0$

We need to modify $N_i(\xi,\eta)$ $i=1\ldots4$ to adapt to central node $(\xi=0, \eta=0)$

$\xi=0, \eta=0 \rightarrow N_1 = 0$

$N_1 = \frac{1}{4} + \alpha \rightarrow \alpha = -\frac{1}{4}$

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta) + \alpha\underbrace{(1-\xi^2)(1-\eta^2)}_{N_5^e(\xi,\eta)}$$

$$N_2^e = \frac{1}{4}(1+\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2)$$

$\vdots$

in general:

corners $\underset{i=1\ldots4}{N_i^e}(\xi,\eta) = \frac{1}{4}(1-\xi_i)(1-\eta_i) - \frac{1}{4}(1-\xi_i^2)(1-\eta_i^2)$

$n=i$	1
$n\neq i$	0

central $N_5^e(\xi,\eta) = (1-\xi^2)(1-\eta^2)$

$$N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}(1-\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta) - \cancel{\frac{4}{4}(1-\xi^2)(1-\eta^2)}$$

$$+ \cancel{(1-\xi^2)(1-\eta^2)} = \frac{1}{4}\left[\textcircled{1} - \cancel{\eta} - \cancel{\xi} + \cancel{\xi\eta} + \textcircled{1} - \cancel{\eta} + \cancel{\xi} - \xi\eta + 1 + \eta + \cancel{\xi} + \xi\eta + 1 + \eta - \cancel{\xi} - \cancel{\xi\eta}\right]$$

$$= \frac{1}{4}\cdot 4 = \textcircled{1}$$

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Problem 5.3 "Convergence requirements"

Which Minimum integration rules of Gauss-product type gives a "rank sufficient" stiffness matrix for these elements:

1. 8-node Hexahedron
2. 20-node Hexahedron
3. 27-node Hexahedron
4. 64-node Hexahedron

$$\boxed{\overset{\text{element DoF}}{n_F} = \overset{\text{n° nodes}}{n} \cdot \overset{\text{dof}}{3}}$$

independent rigid body modes

$$\boxed{n_r = 6}$$ (3 rotational, 3 translational) 3D

$$\boxed{n_E = 6}$$ order of stress-strain matrix

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon & \cdots & \varepsilon \\ & & & & \vdots \\ & & & & \varepsilon_{\alpha} \end{pmatrix}$$

$$r = \min(n_F - n_R,\ n_E \cdot n_G)$$

n_G n° of gauss points

for a "rank sufficient stiffness matrix (and element), $\boxed{n_E \cdot n_G \geq n_F - n_R}$

$$\boxed{6 n_G \geq 3n - 6}$$

1. 8 node ($n=8$) $\rightarrow n_G \geq \frac{24-6}{6} \rightarrow \boxed{n_G = 3}$ ← minimum integration rules of a Gauss-product type for element to be "rank sufficient"
2. 20 node ($n=20$) $\rightarrow \boxed{n_G = 9}$
3. 27 node ($n=27$) $\rightarrow n_G \geq 12'5 \rightarrow \boxed{n_G = 13}$
4. 64 node ($n=64$) $\rightarrow n_G \geq 31 \rightarrow \boxed{n_G = 31}$