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Assignment 5.1

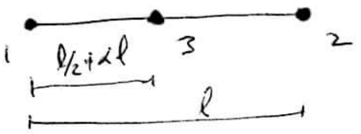
The isoparametric definition of the straight-node bar in its local system $\bar{x}$ is,

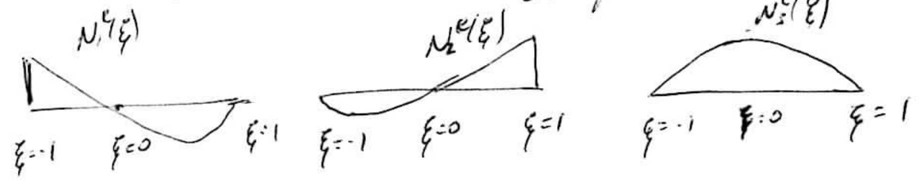
$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = l$, $\bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$

Show that the minimum α (minimal in absolute value sense) for which $J = \frac{d\bar{x}}{\partial\xi}$ vanishes at point in the element are $\pm\frac{1}{4}$ (the quarter point. Interpret this result as singularity by showing that the axial strain becomes infinite at an end point.





$N_1^e(\xi) = -\frac{1}{2}\xi(1-\xi)$ $\qquad \frac{\partial N_1}{\partial\xi} = -\frac{1}{2} + \xi$

$N_2^e(\xi) = \frac{1}{2}\xi(1+\xi)$ $\qquad \frac{\partial N_2}{\partial\xi} = \frac{1}{2} + \xi$

$N_3^e(\xi) = 1 - \xi^2$ $\qquad \frac{\partial N_3}{\partial\xi} = -2\xi$

$$\bar{x} = \bar{x}_1 N_1(\xi) + \bar{x}_2 N_2(\xi) + \bar{x}_3 N_3(\xi) = l N_2(\xi) + (l/2 + \alpha l) N_3(\xi)$$

$$J = |J| = \frac{\partial\bar{x}}{\partial\xi} = l\frac{\partial N_2}{\partial\xi}(\xi) + (l/2 + \alpha l)\frac{\partial N_3(\xi)}{\partial\xi} = l\left(\frac{1}{2} + \cancel{\xi} - \cancel{\xi} - 2\alpha\xi\right)$$

$|J| = l\left(\frac{1}{2} - 2\alpha\xi\right) > 0$ Jacobian Positiveness

$\frac{1}{2} - 2\alpha\xi > 0$

if $\xi = 1$ $\quad -2\alpha > -\frac{1}{2}$ $\quad$ if $\xi = -1$ $\quad 2\alpha > -\frac{1}{2}$

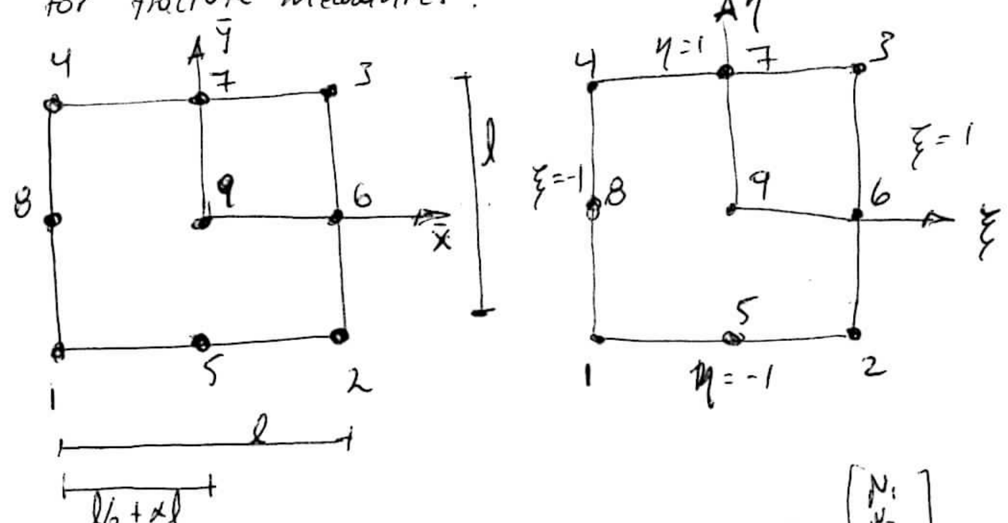
$\alpha < \frac{1}{4}$ $\qquad \alpha > -\frac{1}{4}$

$|\alpha| < \frac{1}{4}$

Assignment 5.2

Extend the result obtained from previous excercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7, 8 are at the mid point of the sides 1-2, 2-3, 3-4 and 4-1, respectively, and 9 at the center of the square

Move node 5 tangetially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.



$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \bar{x}_6 & \bar{x}_7 & \bar{x}_8 & \bar{x}_9 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 & \bar{y}_6 & \bar{y}_7 & \bar{y}_8 & \bar{y}_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \cdot \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & l & l & 0 & l/2+\alpha l & l & l/2 & 0 & l/2 \\ 0 & 0 & l & l & 0 & l/2 & l & l/2 & l/2 \end{bmatrix} \cdot \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \end{bmatrix} = \begin{bmatrix} lN_2 + lN_3 + (l/2+\alpha l)N_5 + lN_6 + l/2 N_7 + l/2 N_9 \\ lN_3 + lN_4 + l/2 N_6 + lN_7 + l/2 N_8 + l/2 N_9 \end{bmatrix}$$

At 2 $\xi = 1$ $\eta = -1$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta$	$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(\eta-\eta^2-2\xi\eta+2\xi\eta^2)$	$\frac{\partial N_1}{\partial \xi} = \frac{1}{2}$
$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$	$\frac{\partial N_2}{\partial \xi} = -\frac{1}{4}(\eta-\eta^2+2\xi\eta-2\xi\eta^2)$	$\frac{\partial N_2}{\partial \xi} = \frac{3}{2}$
$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta$	$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(\eta+\eta^2+2\xi\eta+2\xi\eta^2)$	$\frac{\partial N_3}{\partial \xi} = 0$
$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta$	$\frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(\eta+\eta^2-2\xi\eta-2\xi\eta^2)$	$\frac{\partial N_4}{\partial \xi} = 0$
$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$	$\frac{\partial N_5}{\partial \xi} = -\frac{1}{2}(-2\xi\eta+2\xi\eta^2)$	$\frac{\partial N_5}{\partial \xi} = -2$
$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$	$\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(1-\eta^2+2\xi-2\xi\eta^2)$	$\frac{\partial N_6}{\partial \xi} = 0$
$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)\eta$	$\frac{\partial N_7}{\partial \xi} = \frac{1}{2}(-2\xi\eta-2\xi\eta^2)$	$\frac{\partial N_7}{\partial \xi} = 0$
$N_8 = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi$	$\frac{\partial N_8}{\partial \xi} = -\frac{1}{2}(1-\eta^2-2\xi+2\xi\eta^2)$	$\frac{\partial N_8}{\partial \xi} = 0$
$N_9 = (1-\xi^2)(1-\eta^2)$	$\frac{\partial N_9}{\partial \xi} = -2\xi+2\xi\eta^2$	$\frac{\partial N_9}{\partial \xi} = 0$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(\xi - 2\xi\eta - \xi^2 + 2\xi^2\eta)$$
$$\frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(\xi - 2\xi\eta + \xi^2 - 2\xi^2\eta)$$
$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(\xi + 2\xi\eta + \xi^2 + 2\xi^2\eta)$$
$$\frac{\partial N_4}{\partial \eta} = -\frac{1}{4}(\xi + 2\xi\eta - \xi^2 - 2\xi^2\eta)$$
$$\frac{\partial N_5}{\partial \eta} = -\frac{1}{2}(1 - 2\eta - \xi^2 + 2\xi^2\eta)$$
$$\frac{\partial N_6}{\partial \eta} = \frac{1}{2}(-2\xi\eta - \xi^2\eta)$$
$$\frac{\partial N_7}{\partial \eta} = \frac{1}{2}(1 + 2\eta - \xi^2 - 2\xi^2\eta)$$
$$\frac{\partial N_8}{\partial \eta} = -\frac{1}{2}(-2\xi\eta + 2\xi^2\eta)$$
$$\frac{\partial N_9}{\partial \eta} = -2\eta + 2\xi^2\eta$$

At 2 $\xi = 1$ $\eta = -1$

$$\frac{\partial N_1}{\partial \eta} = 0$$
$$\frac{\partial N_2}{\partial \eta} = -\frac{3}{2}$$
$$\frac{\partial N_3}{\partial \eta} = -\frac{1}{2}$$
$$\frac{\partial N_4}{\partial \eta} = 0$$
$$\frac{\partial N_5}{\partial \eta} = 0$$
$$\frac{\partial N_6}{\partial \eta} = 2$$
$$\frac{\partial N_7}{\partial \eta} = 0$$
$$\frac{\partial N_8}{\partial \eta} = 0$$
$$\frac{\partial N_9}{\partial \eta} = 0$$

$$J = \begin{pmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} \end{pmatrix} = l \begin{bmatrix} \frac{\partial N_2}{\partial \xi} + \frac{\partial N_3}{\partial \xi} + \frac{1}{2}\frac{\partial N_5}{\partial \xi} + \alpha\frac{\partial N_5}{\partial \xi} + \frac{\partial N_6}{\partial \xi} + \frac{1}{2}\frac{\partial N_7}{\partial \xi} + \frac{1}{2}\frac{\partial N_9}{\partial \xi} & \frac{\partial N_3}{\partial \xi} + \frac{\partial N_4}{\partial \xi} + \frac{1}{2}\frac{\partial N_6}{\partial \xi} + \frac{\partial N_7}{\partial \xi} + \frac{1}{2}\frac{\partial N_8}{\partial \xi} + \frac{1}{2}\frac{\partial N_9}{\partial \xi} \\ \frac{\partial N_2}{\partial \eta} + \frac{\partial N_3}{\partial \eta} + \frac{1}{2}\frac{\partial N_5}{\partial \eta} + \alpha\frac{\partial N_5}{\partial \eta} + \frac{\partial N_6}{\partial \eta} + \frac{1}{2}\frac{\partial N_7}{\partial \eta} + \frac{1}{2}\frac{\partial N_9}{\partial \eta} & \frac{\partial N_3}{\partial \eta} + \frac{\partial N_4}{\partial \eta} + \frac{1}{2}\frac{\partial N_6}{\partial \eta} + \frac{\partial N_7}{\partial \eta} + \frac{1}{2}\frac{\partial N_8}{\partial \eta} + \frac{1}{2}\frac{\partial N_9}{\partial \eta} \end{bmatrix}$$

$$J = l \begin{pmatrix} \frac{3}{2} - 1 - 2\alpha & 0 \\ -\frac{3}{2} - \frac{1}{2} + 2 & -\frac{1}{2} + 1 \end{pmatrix} = l \begin{bmatrix} \frac{1}{2} - 2\alpha & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$\det J = l^2\left(\frac{1}{4} - \alpha\right) > 0$ Jacobian Positiveness

$$\frac{1}{4} - \alpha > 0$$
$$-\alpha > -\frac{1}{4}$$
$$\boxed{\alpha < \frac{1}{4}}$$