Computational Structural Mechanics and Dynamics **March 12, 2019**
Master of Science in Computational Mechanics
Universitat Politècnica de Catalunya **Agustina FELIPE RAMUDO**

Assignment 5

5.1 The isoparametric definition of the straightnode bar element in its local system $\underline{x}$ is,

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{1}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = l$, $\bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum a (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm\frac{1}{4}$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

For the given element, the shape functions are:

$$N_1 = \frac{1}{2}\xi(\xi - 1)$$
$$N_2 = \frac{1}{2}\xi(\xi + 1)$$
$$N_3 = 1 - \xi^2$$

And using the first expresion:

$$\begin{aligned} x &= \bar{x}_1 N_1 + \bar{x}_2 N_2 + \bar{x}_3 N_3 \\ &= 0 \cdot N_1 + l \cdot \frac{1}{2}\xi(\xi + 1) + (\frac{l}{2} + \alpha l) \cdot (1 - \xi^2) \\ &= \frac{l}{2}\xi(\xi + 1) + (\frac{l}{2} + \alpha l) \cdot (1 - \xi^2) \end{aligned}$$

Then, the Jacobian :

$$J = \frac{d\bar{x}}{d\xi} = \frac{l}{2}(\xi + 1) + \frac{l}{2}\xi + (\frac{l}{2} + \alpha l)(-2\xi) = \frac{l}{2} - 2\alpha l \xi \tag{2}$$

which vanishes for $\alpha = \pm 1/4$ and $\xi \neq 0$, i.e. at the end nodes.

Again, using the first expression, the displacement vector is:

$$u = u_1 N_1 + u_2 N_2 + u_3 N_3 \tag{3}$$

Considering $\varepsilon = \frac{du}{dx}$, it is possible to obtain:

$$\varepsilon = u_1 \frac{dN_1}{dx} + u_2 \frac{dN_2}{dx} + u_3 \frac{dN_3}{dx} = u_1 \frac{dN_1}{d\xi} \cdot \frac{d\xi}{dx} + u_2 \frac{dN_2}{d\xi} \cdot \frac{d\xi}{dx} + u_3 \frac{dN_3}{d\xi} \cdot \frac{d\xi}{dx} \tag{4}$$

Since $\frac{d\xi}{dx} = J^{-1}$ and $J = 0$ for $\alpha = \pm 1/4$ at the end points, the strain value becomes infinite.

5.2 Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 12, 23, 34 and 41, respectively, and 9 at the center of the square.
Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of singular elements for fracture mechanics.

For the given element, the shapes functions can be found using the line-product method:

- $N_1 = \frac{1}{4}\xi\eta(\xi - 1)(\eta - 1)$
- $N_2 = \frac{1}{4}\xi\eta(\xi + 1)(\eta - 1)$
- $N_3 = \frac{1}{4}\xi\eta(\xi + 1)(\eta + 1)$
- $N_4 = \frac{1}{4}\xi\eta(\xi - 1)(\eta + 1)$
- $N_5 = \frac{1}{2}\eta(1 - \xi^2)(\eta - 1)$
- $N_6 = \frac{1}{2}\xi(\xi + 1)(1 - \eta^2)$
- $N_7 = \frac{1}{2}\eta(1 - \xi^2)(\eta + 1)$
- $N_8 = \frac{1}{2}\xi(\xi - 1)(1 - \eta^2)$
- $N_9 = (1 - \xi^2)(1 - \eta^2)$

The geometric coordinates:

$$x = \sum_{i=1}^{9} x_i N_i \tag{5}$$

$$y = \sum_{i=1}^{9} y_i N_i \tag{6}$$

The Jacobian matrix $\mathbf{J}$ for the given problem is defined by the following expression:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{9} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{9} \frac{\partial N_i}{\partial \xi} y_i \\ \\ \sum_{i=1}^{9} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{9} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{7}$$

The partial derivatives of the shape functions have the following form:

- $\frac{\partial N_1}{\partial \xi} = \frac{1}{4}\eta(2\xi - 1)(\eta - 1)$
- $\frac{\partial N_1}{\partial \eta} = \frac{1}{4}\xi(\xi - 1)(2\eta - 1)$
- $\frac{\partial N_2}{\partial \xi} = \frac{1}{4}\eta(2\xi + 1)(\eta - 1)$
- $\frac{\partial N_2}{\partial \eta} = \frac{1}{4}\xi(\xi + 1)(2\eta - 1)$
- $\frac{\partial N_3}{\partial \xi} = \frac{1}{4}\eta(2\xi + 1)(\eta + 1)$
- $\frac{\partial N_3}{\partial \eta} = \frac{1}{4}\xi(\xi + 1)(2\eta + 1)$
- $\frac{\partial N_4}{\partial \xi} = \frac{1}{4}\eta(2\xi - 1)(\eta + 1)$
- $\frac{\partial N_4}{\partial \eta} = \frac{1}{4}\xi(\xi - 1)(2\eta + 1)$
- $\frac{\partial N_5}{\partial \xi} = -\xi\eta(\eta - 1)$
- $\frac{\partial N_5}{\partial \eta} = \frac{1}{2}(1 - \xi^2)(2\eta - 1)$
- $\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(2\xi + 1)(1 - \eta^2)$

- $\dfrac{\partial N_6}{\partial \eta} = -\xi\eta(\xi + 1)$

- $\dfrac{\partial N_7}{\partial \xi} = -\xi\eta(\eta + 1)$

- $\dfrac{\partial N_7}{\partial \eta} = \dfrac{1}{2}(1 - \xi^2)(2\eta + 1)$

- $\dfrac{\partial N_8}{\partial \xi} = \dfrac{1}{2}(2\xi - 1)(1 - \eta^2)$

- $\dfrac{\partial N_8}{\partial \eta} = -\xi\eta(\xi - 1)$

- $\dfrac{\partial N_9}{\partial \xi} = -2\xi(1 - \eta^2)$

- $\dfrac{\partial N_9}{\partial \eta} = -2\eta(1 - \xi^2)$

For node 2 ($\xi = 1$, $\eta = -1$), the Jacobian reduces to:

$$\mathbf{J}(\mathbf{1}, -\mathbf{1}) = \begin{bmatrix} \frac{l}{2} - 2a & 0 \\ 0 & \frac{l}{2} \end{bmatrix}$$

The determinant of the Jacobian vanishes for the following value of α:

$$\begin{gathered} \mid J \mid = 0 \\ \frac{l^2}{4} - la = 0 \\ a = \frac{l}{4} \end{gathered}$$