Master Of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

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Assignment 5

Assignment 5.1

The isoparametric definition of the straightnode bar element in its local system $\underline{x}$ is,

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{1}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = l$, $\bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum a (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm 1/4$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

Answer

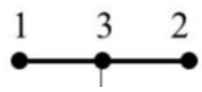

Figure 1: 1D Isoparametric element.

Axial displacement:

$$u = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3 \tag{2}$$

and the x coordinate of any point within the element:

$$x = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3 \tag{3}$$

The shape functions for the element:

$$\begin{aligned} N_1 &= \frac{1}{2}\xi(\xi - 1) \\ N_2 &= \frac{1}{2}\xi(\xi + 1) \\ N_3 &= 1 - \xi^2 \end{aligned} \tag{4}$$

The axial strain is given by:

$$\epsilon = \frac{du}{dx} = \sum_{i=1}^{3} \frac{dN_i}{d\xi} u_i = \begin{bmatrix} \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x}, \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x}, \frac{\partial N_3}{\partial \xi}\frac{d\xi}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{5}$$

taking derivative of shape function with respect to ξ:

$$\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2}, \qquad \frac{\partial N_2}{\partial \xi} = \xi + \frac{1}{2}, \qquad \frac{\partial N_3}{\partial \xi} = -2\xi$$

taking derivative of x coordinate:

$$\begin{aligned}\frac{\partial x}{\partial \xi} &= \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 \\ &= \left(\xi - \frac{1}{2}\right)(0) + \left(\xi + \frac{1}{2}\right)(l) + (-2\xi)\left(\frac{l}{2} + \alpha l\right) \\ &= \frac{l}{2} - 2\alpha l\xi = \frac{1}{2}(l - 4\alpha l\xi)\end{aligned}$$

Relationship between dx and $d\xi$ in terms of the three nodal coordinates can be given as:

$$\frac{\partial \xi}{\partial x} = \frac{2}{l - 4\alpha l\xi} \tag{6}$$

The strains at the first and second node are:

$$\frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x} = \left(\xi - \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l\xi}\right) = \left((-1) - \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l(-1)}\right) = \frac{-3}{l(1+4\alpha)}$$

$$\frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x} = \left(\xi + \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l\xi}\right) = \left((1) + \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l(1)}\right) = \frac{3}{l(1-4\alpha)}$$

For values,
Node 1: $\xi = -1$ with $\alpha = -1/4$, Node 2: $\xi = 1$ with $\alpha = 1/4$

Conclusion: At $\alpha = \pm 1/4$, the strain value becomes infinite.

Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6 ,7 8 are at the midpoint of the sides 1-2, 2-3, 3-4 and 4-1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of singular elements for fracture mechanics.

Answer
Considering a plane stress quadrilateral element with 9 nodes

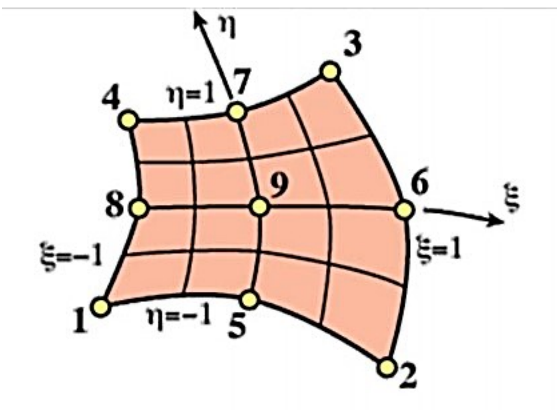

Figure 2: 2D Isoparametric quadrilateral element.

Formulation of the approximate solution,

$$x = \sum_{i=1}^{n} x_i N_i \qquad y = \sum_{i=1}^{n} y_i N_i$$

The shape functions of nodes :

- **Node 1:** $N_1 = \frac{1}{4}\xi\eta(\xi - 1)(\eta - 1)$
- **Node 2:** $N_2 = \frac{1}{4}\xi\eta(\xi + 1)(\eta - 1)$
- **Node 3:** $N_3 = \frac{1}{4}\xi\eta(\xi + 1)(\eta + 1)$
- **Node 4:** $N_4 = \frac{1}{4}\xi\eta(\xi - 1)(\eta + 1)$
- **Node 5:** $N_5 = \frac{1}{2}\eta(1 - \xi^2)(\eta - 1)$
- **Node 6:** $N_6 = \frac{1}{2}\xi(\xi + 1)(1 - \eta^2)$
- **Node 7:** $N_7 = \frac{1}{2}\eta(1 - \xi^2)(\eta + 1)$
- **Node 8:** $N_8 = \frac{1}{2}\xi(\xi - 1)(1 - \eta^2)$
- **Node 9:** $N_9 = (1 - \xi^2)(1 - \eta^2)$

Differentiating with respect to the quadrilateral coordinates:

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \xi}, \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \xi}, \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \eta}, \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \eta},$$

The Jacobian matrix can be given by,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{7}$$

Components of Jacobian matrix:

- **Node 1:** $\frac{\partial N_1}{\partial \xi} = \frac{1}{4}\eta(2\xi - 1)(\eta - 1)$ $\quad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}\xi(\xi - 1)(2\eta - 1)$
- **Node 2:** $\frac{\partial N_2}{\partial \xi} = \frac{1}{4}\eta(2\xi + 1)(\eta - 1)$ $\quad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}\xi(\xi + 1)(2\eta - 1)$
- **Node 3:** $\frac{\partial N_3}{\partial \xi} = \frac{1}{4}\eta(2\xi + 1)(\eta + 1)$ $\quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}\xi(\xi + 1)(2\eta + 1)$
- **Node 4:** $\frac{\partial N_4}{\partial \xi} = \frac{1}{4}\eta(2\xi - 1)(\eta + 1)$ $\quad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}\xi(\xi - 1)(2\eta + 1)$
- **Node 5:** $\frac{\partial N_5}{\partial \xi} = -\xi\eta(\eta - 1)$ $\quad \frac{\partial N_5}{\partial \eta} = \frac{1}{2}(1 - \xi^2)(2\eta - 1)$
- **Node 6:** $\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(2\xi + 1)(1 - \eta^2)$ $\quad \frac{\partial N_6}{\partial \eta} = -\xi\eta(\xi + 1)$
- **Node 7:** $\frac{\partial N_7}{\partial \xi} = -\xi\eta(\eta + 1)$ $\quad \frac{\partial N_7}{\partial \eta} = \frac{1}{2}(1 - \xi^2)(2\eta + 1)$
- **Node 8:** $\frac{\partial N_8}{\partial \xi} = \frac{1}{2}(2\xi - 1)(1 - \eta^2)$ $\quad \frac{\partial N_8}{\partial \eta} = -\xi\eta(\xi - 1)$
- **Node 9:** $\frac{\partial N_9}{\partial \xi} = -2\xi(1 - \eta^2)$ $\quad \frac{\partial N_9}{\partial \eta} = -2\eta(1 - \xi^2)$

For node 2, $(\xi, \eta) = (1, -1)$:

$$\mathbf{J}(\xi, \eta) = \begin{bmatrix} \frac{l}{2} - 2a & 0 \\ 0 & \frac{l}{2} \end{bmatrix}$$

The determinant of the Jacobian:

$$| J |= 0$$
$$\frac{l^2}{4} - l\alpha = 0$$
$$\alpha = \frac{l}{4}$$

Which is similar to to 1D element.