Computational Structural Mechanics and Dynamics
Assignment 5

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Contents

1 Problem Description

1.1 Assignment 5.1

The isoparametric definition of the straight-node bar element in its local system $\overline{x}$ is,

$$\begin{bmatrix} 1 \\ \overline{x} \\ \overline{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \overline{x}_1 & \overline{x}_2 & \overline{x}_3 \\ \overline{u}_1 & \overline{u}_2 & \overline{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{1}$$

Here ξ is the isoparametric coordinate that takes the values -1,1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\overline{x}_1 = 0$, $\overline{x}_2 = L$, $\overline{x}_3 = \frac{L}{2} + \alpha L$. Here L is the bar length and α is a parameter that characterizes how far node 3 is away from the midpoint location $\overline{x} = \frac{L}{2}$.

Show that the minimum α (minimal in absolute value sense) for which $J = d\overline{x}/d\xi$ vanishes at a point in the element are $\pm\frac{1}{4}$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

1.2 Assignment 5.2

Extend the results obtained from the previous exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7, 8 are at the midpoint of the sides 1-2, 2-3, 3-4 and 4-1, respectively. Node 9 is at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

2 Solution

2.1 Assignment 5.1

For a 3-node **isoparametric** bar element, the node coordinates are defined as:

$$\xi_1 = -1 \qquad \xi_2 = 1 \qquad \xi_3 = 0 \tag{2}$$

Therefore, the shape functions take the form:

$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1) \qquad N_2(\xi) = \frac{1}{2}\xi(\xi + 1) \qquad N_3(\xi) = 1 - \xi^2 \tag{3}$$

and the corresponding derivatives:

$$\frac{dN_1}{\xi} = \xi - \frac{1}{2} \qquad \frac{dN_2}{\xi} = \xi + \frac{1}{2} \qquad \frac{dN_3}{\xi} = -2\xi \tag{4}$$

Now, from equation (1) we know that $\overline{x}$ can be expressed as a function of ξ as:

$$\overline{x} = \sum_{i=1}^{3} \overline{x}_i N_i(\xi) \tag{5}$$

We may then obtain the Jacobian by deriving $\overline{x}$ with respect to ξ

$$J = \frac{d\overline{x}}{d\xi} = \sum_{i=1}^{3} \overline{x}_i \frac{dN_i(\xi)}{d\xi} \tag{6}$$

$$\boxed{J = l\big(\frac{1}{2} - 2\alpha\xi\big)} \tag{7}$$

Now, in order to find the minimum absolute value of α for which the Jacobian vanishes, we force the value of left hand side of equation (7) to zero. Therefore, we must find the value of α for which the following equality is fullfilled.

$$2\alpha\xi = \frac{1}{2} \qquad \rightarrow \qquad \alpha = \frac{1}{4}\frac{1}{\xi} \tag{8}$$

Applying absolute values on both sides of the equation yields:

$$|\alpha| = \frac{1}{4}\left|\frac{1}{\xi}\right| \tag{9}$$

The minimum absolute value of α will correspond to the maximum absolute value of ξ since they are inversely proportional. Taking into account that $-1 \leq \xi \leq 1$, α becomes:

$$|\alpha| = \left|\frac{1}{4}\right| \qquad \rightarrow \qquad \boxed{\alpha = \pm\frac{1}{4}} \tag{10}$$

We are now interested in analyzing what happens to the strains when α takes this value. Axial strains in 1D are given by:

$$\varepsilon = \frac{d\overline{u}}{d\overline{x}} = \frac{d\overline{u}}{d\xi}\frac{d\xi}{d\overline{x}} = \boxed{J^{-1}\frac{d\overline{u}}{d\xi}} \tag{11}$$

It is evident that when $\alpha = \pm\frac{1}{4}$, the Jacobian is zero and the axial strains become infinite. Therefore, $\alpha = \frac{1}{4}$ is a singularity for this problem. Node 3 must remain within the central third of the element to avoid this.

2.2 Assignment 5.2

The shape functions for an isoparametric formulation of the described element are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \tag{12}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta \qquad N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta \tag{13}$$

$$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi \tag{14}$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)\eta \qquad N_8 = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi \tag{15}$$

$$N_9 = (1-\xi^2)(1-\eta^2) \tag{16}$$

For $-1 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1$

In the physical domain, the coordinates of all 9 nodes are:

Node	1	2	3	4	5	6	7	8	9
x	0	L	L	0	$L/2+\alpha L$	L	L/2	0	L/2
y	0	0	L	L	0	L/2	L	L/2	L/2

Table 1: Node coordinates, 9-node quadrilateral

By extending equation (1), x and y may be approximated as:

$$x = \sum_{i=1}^{9} x_i N_i(\xi) \qquad y = \sum_{i=1}^{9} y_i N_i(\xi) \tag{17}$$

Then, by deriving x and y with respect to ξ and η, the Jacobian in 2D may be computed as:

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} \tag{18}$$

Now, we will compute the Jacobian for the cases $\boldsymbol{\alpha = 0}$ and $\boldsymbol{0 < \alpha < 1/2}$. On the second case, we will seek the value of α for which the Jacobian determinant vanishes at node 2. Appendix A displays the Matlab code implemented for the following computations.

- $\alpha = 0$
  In this case, the coordinates for node 5 become $(\frac{L}{2}, 0)$. Applying equations (18) and (19) yields:

$$x = \frac{L}{2}(\xi + 1) \qquad y = \frac{L}{2}(\eta + 1) \tag{19}$$

$$J = \begin{bmatrix} \frac{L}{2} & 0 \\ \\ 0 & \frac{L}{2} \end{bmatrix} \qquad |J| = \frac{L^2}{4} \tag{20}$$

- $0 < \alpha < 1/2$
  Node 5 is displacing towards node 2. Therefore, the value of α must be positive and it cannot exceed 1/2 for node 5 to stay within the lower edge of the element.

$$x = \frac{\alpha L}{2}(\xi + 1)\left(\eta^2 - \eta + \xi\eta - \xi\eta^2 + \frac{1}{\alpha}\right) \qquad y = \frac{L}{2}(\eta + 1) \tag{21}$$

$$J = \begin{bmatrix} \frac{\alpha L}{2}\left(2\xi\eta - 2\xi\eta^2 + \frac{1}{\alpha}\right) & 0 \\ \\ -\frac{\alpha L}{2}(\xi^2 - 1)(2\eta - 1) & \frac{L}{2} \end{bmatrix} \tag{22}$$

$$|J| = \frac{\alpha L^2}{4}\left(2\xi\eta - 2\xi\eta^2 + \frac{1}{\alpha}\right) \tag{23}$$

Making the left hand side of the equation go to zero yields:

$$\alpha = \frac{1}{2\xi\eta(\eta - 1)} \tag{24}$$

Replacing ξ and η with the coordinates of node 2 (1,-1), α becomes:

$$\boxed{\alpha = \frac{1}{4}} \tag{25}$$

Just like in Assignment 5.1, $\alpha = \frac{1}{4}$ represents a singularity since it makes the Jacobian determinant go to zero. This means that for the 2D quadratic quadrilateral case, the nodes located on the edges of the element must remain within the central third of the element, as they must in the 1D case.

A Appendices

A.1 Matlab Code

```
clear all
clc

% case 1
syms L xi eta
assume(L,'positive')
assume(xi,'real')
assume(eta,'real')
alpha = 0;

% case 2
% syms L xi eta alpha
% assume(L,'positive')
% assume(xi,'real')
% assume(eta,'real')
% assume(alpha,'real')

% NODE COORDINATES IN PHYSICAL COORDINATES
x = [0 L L 0 L/2 + alpha*L L L/2 0 L/2];
y = [0 0 L L 0 L/2 L L/2 L/2];

% SHAPE FUNCTIONS ISO-9
N=[1/4*(1-xi)*(1-eta)*xi*eta;
   -1/4*(1+xi)*(1-eta)*xi*eta;
   1/4*(1+xi)*(1+eta)*xi*eta;
   -1/4*(1-xi)*(1+eta)*xi*eta;
   -1/2*(1-xi^2)*(1-eta)*eta;
   1/2*(1+xi)*(1-eta^2)*xi;
   1/2*(1-xi^2)*(1+eta)*eta;
   -1/2*(1-xi)*(1-eta^2)*xi;
   (1-xi^2)*(1-eta^2)]';

% SHAPE FUNCTIONS DERIVATIVES
 dN_dxi = [1/4*(eta*(1-2*xi)+eta^2*(2*xi-1));
          -1/4*(eta*(1+2*xi)-eta^2*(2*xi+1));
```

```
           1/4*((1+2*xi)*(eta+eta^2));
           -1/4*((1-2*xi)*(eta+eta^2));
           -1/2*(2*xi*eta*(eta-1));
           1/2*(1-eta^2+2*xi-2*xi*eta^2);
           1/2*(-2*xi*eta*(eta+1));
           -1/2*(1-eta^2-2*xi+2*xi*eta^2);
           2*xi*(eta^2-1)]';

dN_deta = [1/4*(xi*(1-2*eta)+xi^2*(2*eta-1));
            -1/4*(xi*(1-2*eta)+xi^2*(1-2*eta));
            1/4*((1+2*eta)*(xi+xi^2));
            -1/4*((1+2*eta)*(xi-xi^2));
            -1/2*(1-2*eta-xi^2+2*xi^2*eta);
            1/2*(-2*xi*eta*(xi+1));
            1/2*(1+2*eta-xi^2-2*xi^2*eta);
            -1/2*(2*xi*eta*(xi-1));
            2*eta*(xi^2-1)]';

% x = x_i*N_i
x_sol = simplify(sum(simplify(x.*N)))
y_sol = simplify(sum(simplify(y.*N)))

dx_dxi = simplify(sum(simplify(x.*dN_dxi)));
dx_deta = simplify(sum(simplify(x.*dN_deta)));
dy_dxi = simplify(sum(simplify(y.*dN_dxi)));
dy_deta = simplify(sum(simplify(y.*dN_deta)));

% Jacobian
J = [dx_dxi dy_dxi;
     dx_deta dy_deta];

Jac = det(J)
```