Computational Structural Mechanics and Dynamics

As5 Convergence requirements

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Assignment 5.1

The isoparametric definition of the straight-node bar element in its local system $\underline{\mathbf{x}}$ **is,**

$$\begin{bmatrix} \mathbf{1} \\ \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_1^e(\xi) \\ \boldsymbol{N}_2^e(\xi) \\ \boldsymbol{N}_2^e(\xi) \end{bmatrix}$$

Here ξ **is the isoparametric coordinate that take the value** $-1, 1$ *and* 0 **at nodes** $1, 2$ *and* 3 **respectively, while** N_1^e, N_2^e *and* N_3^e **are the shape functions for a bar element.**

For simplicity, take $\bar{x}_1 = 0, \bar{x}_2 = l, \bar{x}_3 = \frac{1}{2}l + \alpha l$. **Here l is the bar length and** α **a parameter that characterizes how far node 3 is away from the midpoint location** $\bar{x} = \frac{1}{2}l$

Show that the minimum α **(minimal in absolute value sense) for which** $J = \frac{d\bar{x}}{d\xi}$ **vanishes at a point in the element are** $\pm\frac{1}{4}$ **(the quarter points) interpret the result as a singularity by showing that the axial strain becomes at an end point.**

[Answer]

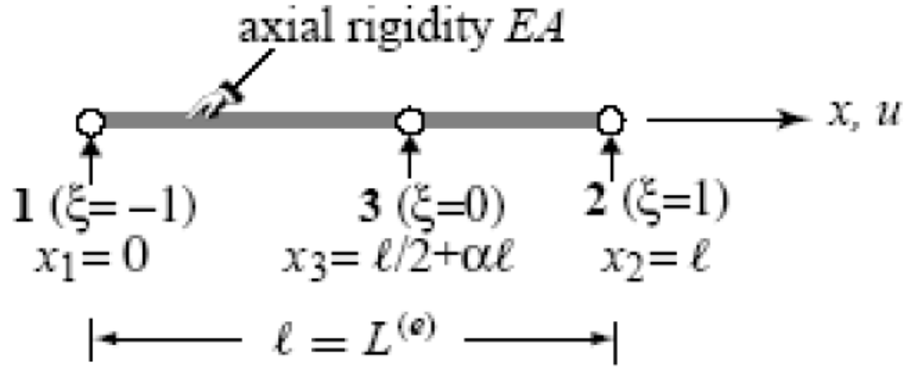


Figure.- The three-node bar element in its local system

$$J = \frac{d\bar{x}}{d\xi} = \sum_{i=1}^{3} \bar{x}_i \frac{dN_i^e}{d\xi}$$

Where,

$$\xi_1 = -1, \xi_2 = 1, \xi_3 = 0$$

$$\bar{x}_1 = 0, \bar{x}_2 = l, \bar{x}_3 = \left(\frac{1}{2} + \alpha\right) l$$

$$N_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{1}{2}\xi(\xi - 1)$$

$$N_2 = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{1}{2}\xi(\xi + 1)$$

$$N_3 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = (1 - \xi^2)$$

So,

$$J = \frac{1}{2}l(1 - 4\alpha\xi)$$

The critical value which we search is when the Jacobian vanishes:

$$J = \frac{1}{2}l(1 - 4\alpha\xi) = 0$$

$$\xi = \frac{1}{4\alpha}$$

$$-1 \leq \xi \leq 1$$

$$-1 \leq \frac{1}{4\alpha} \leq 1$$

$$-1/4 < \alpha < 1/4$$

In the case that

$$|\alpha| = \frac{1}{4}, \xi^* = \pm 1$$

That means that the Jacobian vanishes at one or another end point. In this case, the axial strain become infinite at this point.

That is shown as,

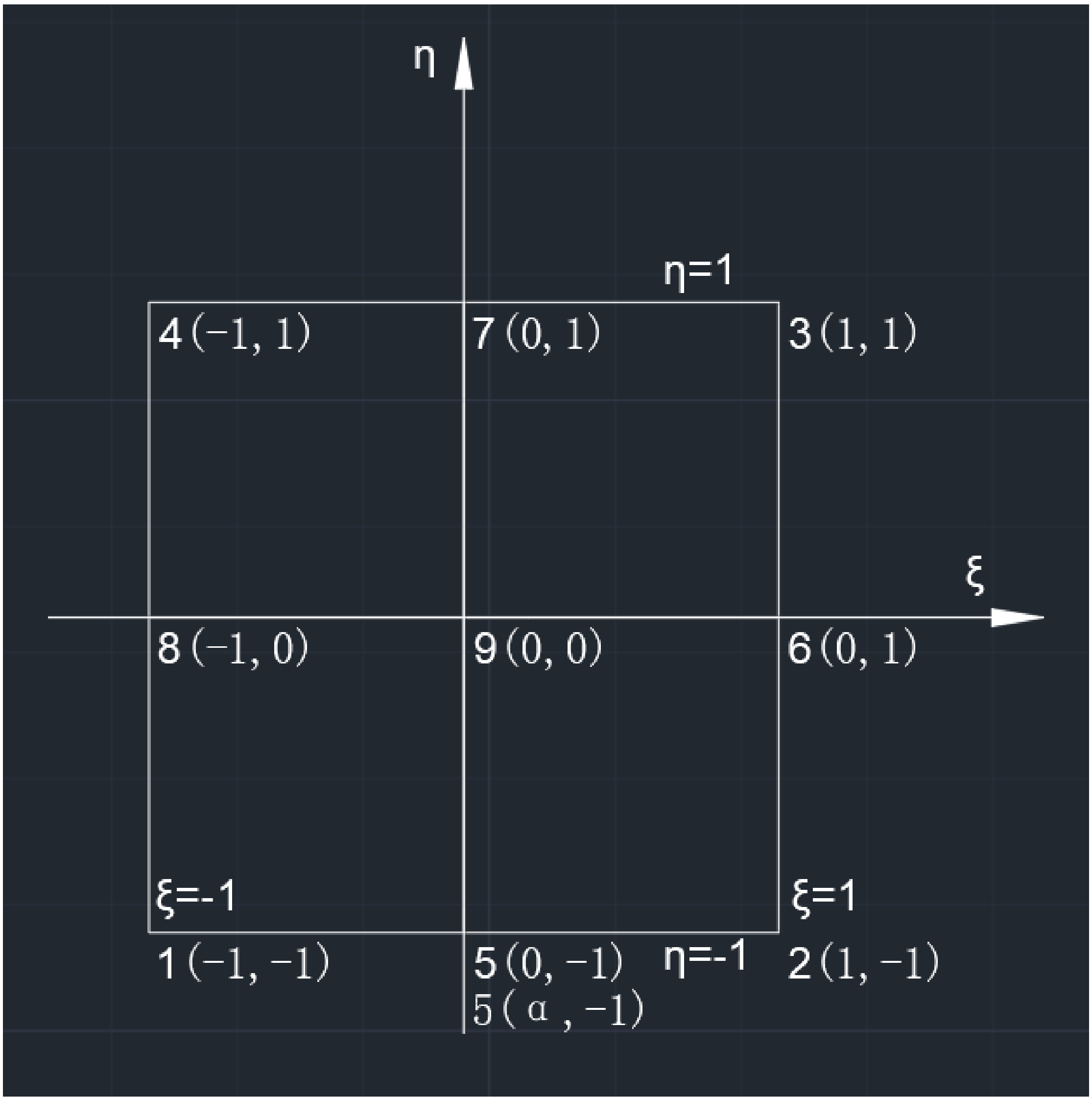
$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = J^{-1}\frac{du}{d\xi}$$

Assignment 5.2

Extend the results obtained from the previous exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7, 8 are at the midpoint of the sides1-2, 2-3, 3-4 and 4-1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanic.

[Answer]



The element shape function of the Lagrange family:

$$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta$$

$$N_2^e = \frac{1}{4}(\xi + 1)(\eta - 1)\xi\eta$$

$$N_3^e = \frac{1}{4}(\xi + 1)(1 + \eta)\xi\eta$$

$$N_4^e = \frac{1}{4}(\xi - 1)(\eta + 1)\xi\eta$$

$$N_5^e = -\frac{1}{2}(\xi - 1)(\xi + 1)\eta(\eta - 1)$$

$$N_6^e = -\frac{1}{2}\xi(\xi + 1)(\eta + 1)(\eta - 1)$$

$$N_7^e = -\frac{1}{2}(\xi - 1)(\xi + 1)\eta(\eta + 1)$$

$$N_8^e = -\frac{1}{2}\xi(\xi - 1)(\eta + 1)(\eta - 1)$$

$$N_9^e = (\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)$$

Then the Jacobian is computed as:

$$J(\xi,\eta) = \sum_{i=1}^{9} \begin{bmatrix} x_i \frac{\partial N_i^e}{\partial \xi} & y_i \frac{\partial N_i^e}{\partial \xi} \\ x_i \frac{\partial N_i^e}{\partial \eta} & y_i \frac{\partial N_i^e}{\partial \eta} \end{bmatrix}$$

$$J_{11}(\xi,\eta) = -\frac{1}{4}(2\xi - 1)(\eta - 1)\eta + \frac{1}{4}(2\xi + 1)(\eta - 1)\eta + \frac{1}{4}(2\xi + 1)(\eta + 1)\eta - \frac{1}{4}(2\xi - 1)(\eta + 1)\eta - \alpha(\eta - 1)\xi\eta - \frac{1}{2}(2\xi + 1)(\eta + 1)(\eta - 1) + \frac{1}{2}(2\xi - 1)(\eta + 1)(\eta - 1)$$

$$J_{12}(\xi,\eta) = -\frac{1}{4}(2\xi - 1)(\eta - 1)\eta - \frac{1}{4}(2\xi + 1)(\eta - 1)\eta + \frac{1}{4}(2\xi + 1)(\eta + 1)\eta + \frac{1}{4}(2\xi - 1)(\eta + 1)\eta + (\eta - 1)\xi\eta - (\eta + 1)\xi\eta$$

$$J_{21}(\xi,\eta) = -\frac{1}{4}(\xi - 1)(2\eta - 1)\xi + \frac{1}{4}(\xi + 1)(2\eta - 1)\xi + \frac{1}{4}(\xi + 1)(2\eta + 1)\xi - \frac{1}{4}(\xi - 1)(2\eta + 1)\xi - \frac{\alpha}{2}(2\eta - 1)(\xi + 1)(\xi - 1) - (\xi + 1)\xi\eta + (\xi - 1)\xi\eta$$

$$J_{22}(\xi,\eta) = -\frac{1}{4}(\xi - 1)\xi(2\eta - 1) - \frac{1}{4}(\xi + 1)\xi(2\eta - 1) + \frac{1}{4}(\xi + 1)\xi(2\eta + 1) + \frac{1}{4}(\xi - 1)\xi(2\eta + 1) + \frac{1}{2}(2\eta - 1)(\xi + 1)(\xi - 1) - \frac{1}{2}(2\eta + 1)(\xi + 1)(\xi - 1)$$

Evaluating the jacobian matrix at the node 2:

$$J(1,-1) = \begin{bmatrix} 1 - 2\alpha & 0 \\ 0 & 1 \end{bmatrix}$$

The determinant is:

$$|J(1,-1)| = 1 - 2\alpha$$

The condition for the jacobian vanishing is $\alpha = \frac{1}{2}$. That is the quarter point.

This condition is the same as per the one-dimensional case calculated in the previous case.