Computational Structural Mechanics and Dynamics
Assignment 5 - Convergence Requirements

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Assignment 5.1:

The isoparametric definition of the straight–node bar element in its local system x is,

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{1}$$

Here ξ is the isoparametric coordinate that takes the values –1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0, \bar{x}_2 = L, \bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum α (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm 1/4$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point

Answer

The element that is going to be analyzed is a 3-node bar. So the element, defined by three nodes, is a quadratic linear element. This means that in order to describe it, three parabolic shape functions are needed. As always, the shape functions have to satisfy $1 = \sum N_i$, easy to see if it is computed the 1^{st} row of system (1). The general way of obtaining the shape function, for n number of nodes is:

$$N_j = \prod_{i=1, i\neq j}^{n} \frac{(\xi_i - \xi)}{(\xi_i - \xi_j)} \tag{2}$$

that in this case will return to us three functions:

$$\begin{aligned} N_1 &= \frac{\xi}{2}(\xi - 1) \\ N_2 &= \frac{\xi}{2}(\xi + 1) \\ N_3 &= 1 - \xi^2 \end{aligned} \tag{3}$$

As can be noticed, these three functions are defined in the isoparametric coordinates and they fulfill the requirements, going to 1 in the respective node and 0 in the others two.

x is defined, from the second row of (1) as:

$$\mathbf{x} = \sum_{i=1}^{3} x_i N_i \tag{4}$$

in which x_i are the cartesian coordinates: $x_1 = 0,\ x_2 = l,\ x_3 = l(\frac{1}{2} + \alpha)$ in this case, and N_i are the shape functions defined in (4).

$$\mathbf{x} = l\frac{\xi}{2}(1 + \xi) + l(\frac{1}{2} + \alpha)(1 - \xi^2) \tag{5}$$

Deriving it respect to ξ, we obtain the Jacobian that, as said before, is a scalar:

$$J = \frac{d\mathbf{x}}{d\xi} = \frac{l}{2} - 2\alpha l\xi = l(\frac{1}{2} - 2\alpha\xi) \tag{6}$$

Taking the equation (6) and imposing equal to 0, it can be find the value of α that makes the Jacobian vanish:

$$\begin{gathered} \frac{1}{2} - 2\alpha\xi = 0 \\ \alpha = \pm\frac{1}{4} \end{gathered} \tag{7}$$

As can be noticed, the Jacobian vanishes when $|\alpha| = 1/4$. It can be seen that is a singularity as if we compute the strain-displacements matrix B, defined as:

$$\begin{gathered} e = Bu^e \\ B = \frac{dN}{dx} \end{gathered} \tag{8}$$

in the isoparametric representation is going to be, for the chain rule

$$B = \frac{dN}{d\xi}\frac{d\xi}{dx} \tag{9}$$

and remembering that the Jacobian is defined from

$$J = \frac{dx}{d\xi}$$

it means that the matrix B takes the following form, from (9)

$$B = \frac{dN}{d\xi}J^{-1} \tag{10}$$

and if $J = 0$ it can be clearly seen from (8) that the axial strain becomes infinite.

Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

Answer

A 9-node plane stress element can be represented as following

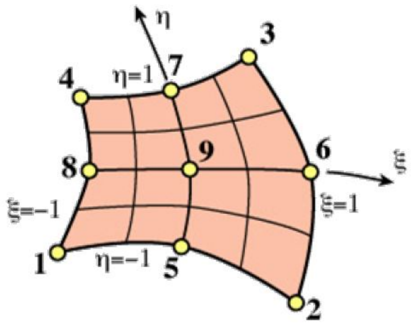

Figure 1: The 9-node element

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In an isoparametric representation, the shape functions will be 0, one for each node, being one in the respective node and zero in the others. The shape functions are defined as following:

The ones related to the four corners, are the same as the 4-node square element, multiplying the isoparametric coordinates in order to take the values of 0 in the mid-point nodes.

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta)\xi\eta \\ N_2 &= -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta)\xi\eta \\ N_4 &= -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta \end{aligned} \tag{11}$$

The four nodes in the mid-points are defined as:

$$\begin{aligned} N_5 &= -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \\ N_6 &= \frac{1}{2}(1+\xi)(1-\eta^2)\xi \\ N_7 &= \frac{1}{2}(1-\xi^2)(1+\eta)\eta \\ N_8 &= -\frac{1}{2}(1-\xi)(1-\eta^2)\xi \end{aligned} \tag{12}$$

and the central one is:

$$N_9 = (1-\xi^2)(1-\eta^2) \tag{13}$$

The approach to be used is the same as in the first part of the assignment, but extended in 2-D. From the last row of (1), considering two dimensions coordinates and 9 shape functions, the two displacements vectors $u, v = u, v(\xi;\eta)$ can be defined as:

$$\begin{aligned} \mathbf{u}(\xi;\eta) &= \sum_{i=1}^{9} u_i N_i(\xi;\eta) \\ \mathbf{v}(\xi;\eta) &= \sum_{i=1}^{9} v_i N_i(\xi;\eta) \end{aligned} \tag{14}$$

and the x, y description of the system as well:

$$\mathbf{x}(\xi;\eta) = \sum_{i=1}^{9} x_i N_i(\xi;\eta)$$
$$\mathbf{y}(\xi;\eta) = \sum_{i=1}^{9} y_i N_i(\xi;\eta) \tag{15}$$

While computing the derivative respect to the cartesian coordinates, it has to be applied the chain rule:

$$\begin{bmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{16}$$

and the first part of the right hand side of the (16) is the Jacobian J. So, when computing $\partial/\partial x$ and $\partial/\partial y$, the expression is:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\eta} \end{bmatrix} \tag{17}$$

In order to describe the physic asked in the assignment, going from a regular square to a different system, sliding the 5^{th} point towards the 2^{nd}, the Jacobian matrix has to be computed for the 9 shape functions, and is defined as:

$$J = \begin{bmatrix} \partial N_i/\partial\xi \\ \partial N_1/\partial\eta \end{bmatrix} \begin{bmatrix} x_i & y_i \end{bmatrix} \tag{18}$$

considering it for 9 nodes it will result in a $2x2$ matrix defined by:

$$J = \begin{bmatrix} \sum_{i=1}^{9} x_i \partial N_i/\partial\xi & \sum_{i=1}^{9} y_i \partial N_i/\partial\xi \\ \sum_{i=1}^{9} x_i \partial N_i/\partial\eta & \sum_{i=1}^{9} y_i \partial N_i/\partial\eta \end{bmatrix} \tag{19}$$

To work in the isoparametric system, all the conditions and expressions needed have been defined in the previous considerations. As can be seen from (16), to complete the conditions needed in order to see what happens if the node 5 collapses forward the node 2, the Jacobian has to be computed and to do so, all the partial derivatives of the shape functions. Considering that the origin of the reference system (x, y) is in the node 1, the coordinates of each node are:

$$\begin{aligned} X_i &= (x_i; y_i) \\ X_1 &= (0; 0) \\ X_2 &= (L; 0) \\ X_3 &= (L; L) \\ X_4 &= (0; L) \\ X_5 &= (L/2 + \alpha L; 0) \\ X_6 &= (L; L/2) \\ X_7 &= (L/2; L) \\ X_8 &= (0; L/2) \\ X_9 &= (L/2; L/2) \end{aligned} \tag{20}$$

in which $-1/2 < \alpha < 1/2$ and initially $\alpha = 0$ so the element is a perfect square. The ones in the isoparametric system are shown in figure 1. Defining the derivative of the shape functions respect to ξ are:

$$\begin{aligned}
\frac{\partial N_1}{\partial \xi} &= \frac{\eta}{4}(1-\eta)(1-2\xi) \\
\frac{\partial N_2}{\partial \xi} &= \frac{\eta}{4}(\eta-1) \\
\frac{\partial N_3}{\partial \xi} &= \frac{\eta}{4}(1+\eta)(1+2\xi) \\
\frac{\partial N_4}{\partial \xi} &= \frac{\eta}{4}(1+\eta)(2\xi-1) \\
\frac{\partial N_5}{\partial \xi} &= \eta\xi(1-\eta) \\
\frac{\partial N_6}{\partial \xi} &= \frac{1-\eta^2}{2}(1+2\xi) \\
\frac{\partial N_7}{\partial \xi} &= -\eta\xi(1+\eta) \\
\frac{\partial N_8}{\partial \xi} &= \frac{1-\eta^2}{2}(2\xi-1) \\
\frac{\partial N_9}{\partial \xi} &= 2\xi(\eta^2-1)
\end{aligned} \tag{21}$$

and respect to η:

$$\begin{aligned}
\frac{\partial N_1}{\partial \eta} &= \frac{\xi}{4}(1-\xi)(1-2\eta) \\
\frac{\partial N_2}{\partial \eta} &= \frac{\xi}{4}(1+\xi)(2\eta-1) \\
\frac{\partial N_3}{\partial \eta} &= \frac{\xi}{4}(1+\xi)(1+2\eta) \\
\frac{\partial N_4}{\partial \eta} &= \frac{\xi}{4}(1-\xi)(-1-2\eta) \\
\frac{\partial N_5}{\partial \eta} &= \frac{1-\xi^2}{2}(2\eta-1) \\
\frac{\partial N_6}{\partial \eta} &= -\xi\eta(1+\xi) \\
\frac{\partial N_7}{\partial \eta} &= \frac{1-\xi^2}{2}(1+2\eta) \\
\frac{\partial N_8}{\partial \eta} &= \eta\xi(1-\xi) \\
\frac{\partial N_9}{\partial \eta} &= 2\eta(\xi^2-1)
\end{aligned} \tag{22}$$

Plugging the values of (20), (21) and (22) in the definition of (19), the Jacobian for the node 2 can be computed plugging the isoparametric coordinates of node 2 $(\xi;\eta) = (1,-1)$. As can be noticed, for those iosparametric coordinates of the node 2, $\partial/\partial\xi$ of the shape functions N_3, N_4, N_6, N_7, N_8 and N_9 is equal to zero. Moreover, considering the zeros of the cartesian coordinates defined at (20), the first two terms of the Jacobian will be:

$$\begin{aligned}
J(1,1) &= L(1/2-2\alpha) \\
J(1,2) &= 0
\end{aligned} \tag{23}$$

Proceeding in the same way as above, for the same isoparametric coordinates, $\partial/\partial\eta$ of the shape functions N_1, N_4, N_5, N_7, N_8 and N_9 is equal to zero. So, considering the coordinates in (20), the second row of the Jacobian takes the form of:

$$\begin{aligned}
J(2,1) &= L(-3/2) + L(-1/2) + 2L = 0 \\
J(2,2) &= L(-1/2) + L = L/2
\end{aligned} \tag{24}$$

The resultant Jacobian for the node 2 is the following:

$$J = \begin{bmatrix} L(\frac{1}{2}-2\alpha) & 0 \\ 0 & \frac{L}{2} \end{bmatrix} \tag{25}$$

Computing the determinant it can be noticed that the results will be the same as in the assignment 5.1

$$|J| = \frac{L^2}{2}(\frac{1}{2} - 2\alpha) \tag{26}$$

and it goes to zero when $\alpha = 1/4$. As can be seen, $\alpha = 1/4$ means that the node 5 is approaching the node 2, as the coordinates of node 5 will be $(\frac{L}{2} + \alpha L; 0) = (\frac{3}{4}L; 0)$.