Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports de Barcelona

MSc Computational Mechanics

Computational Structural Mechanics and dynamics

Beam Theory

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1 Introduction

The bending of beams have been studied in engineering mechanics over the past century. When the length is considerably longer than the width and the thickness, we can treat the structural element as a beam, which help us efficiently calculate the reaction of the element under certain loads. The most well-known beam theories are Euler Bernoulli beam theory and Timoshenko beams theory. Timoshenko beam theory is different from Euler Bernoulli in the way that it takes into account the shear stress in the beam. In the numerical point of view, Timoshenko only requires C_0 continuity for the deflection and rotation. Therefore, the derivation of the linear system is easier. However, if the slenderness ratio $\lambda = \frac{l}{h}$ is very large, this method will suffer from the so-called shear locking effect. It leads to overly stiff numerical solution. Some technique should be applied to deal with this problem. One way is to introduce educed integration. It is achieved by under-integrating the terms in $\mathbf{K}_s^e$ using a quadrature of one order less than is needed for exact integration. In this project, the author implement the reduced shear stiffness matrix and compare the results from Euler Bernoulli element, Timoshenko element using both full integrated shear stiffness matrix and the reduced one.

2 Numerical studies

Program In Mat Lab the Timoshenko 2 Nodes Beam element with reduce integration for the shear stiffness matrix

In the element cycle for calculating elemental stiffness matrices we change the code to:

```
    const = D_matb/len;

   K_b = [ 0 ,  0 , 0 ,  0 ;
           0 ,  1 , 0 , -1 ;
           0 ,  0 , 0 ,  0 ;
           0 , -1 , 0 ,  1 ];

   K_b = K_b * const;

    const = D_mats/len;

   % reduced integration
   K_s = [   1    ,  len/2   ,  -1    ,  len/2   ;
           len/2  , len^2/4  , -len/2 , len^2/4  ;
            -1    , -len/2   ,   1    , -len/2   ;
           len/2  , len^2/4  , -len/2 , len^2/4  ];

   K_s = K_s * const;

   K_elem = K_b + K_s;
```

For stress evaluation we make $gaus1 = gauss2 = 0$ to produce reduced integration.

```
% One gauss point for stress evaluation
```

```
    gaus0 =  0.0;

    bmat_b = [ 0, -1/len , 0, 1/len];

    bmat_s1 = [-1/len,-(1-gaus0)/2, 1/len,-(1+gaus0)/2];

    Str1_g0 = D_matb*(bmat_b *transpose(u_elem));
    Str2_g0 = D_mats*(bmat_s1*transpose(u_elem));
    Strnod(lnods(1),1) = Strnod(lnods(1),1) + Str1_g0;
    Strnod(lnods(2),1) = Strnod(lnods(2),1) + Str1_g0;
    Strnod(lnods(1),2) = Strnod(lnods(1),2) + Str2_g0;
    Strnod(lnods(2),2) = Strnod(lnods(2),2) + Str2_g0;
    Strnod(lnods(1),3) = Strnod(lnods(1),3) + 1;
    Strnod(lnods(2),3) = Strnod(lnods(2),3) + 1;
```

Solve the following problem with a 64 element mesh with:
2 nodes Euler Bernulli element
2 nodes Timoshenko Full Integrate element
2 nodes Timoshenko Reduce Integration element.
Compare maximum displacements, moments and shear for the 3 elements against the a/L relationship

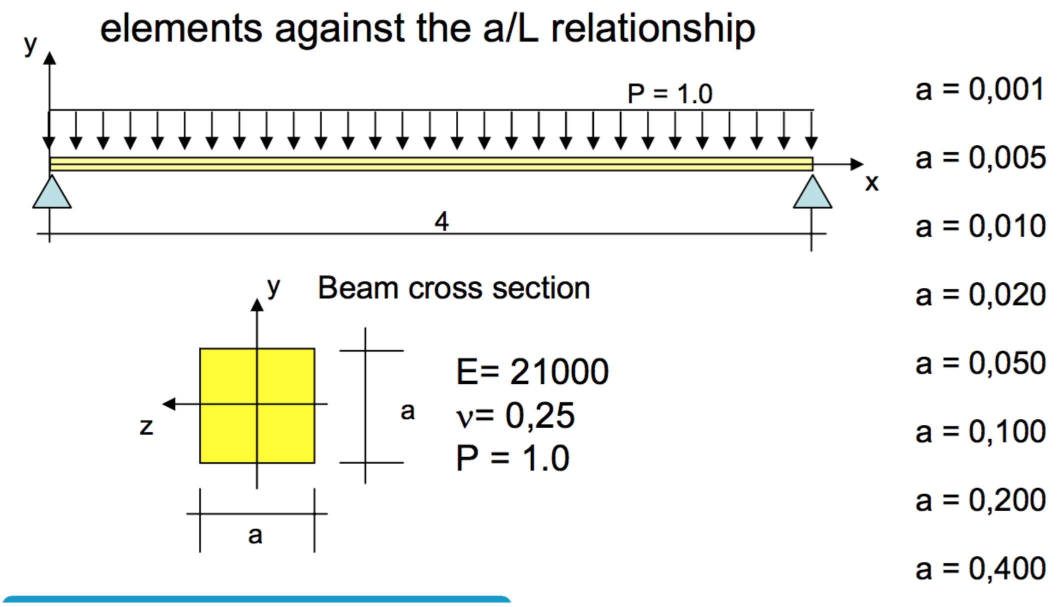


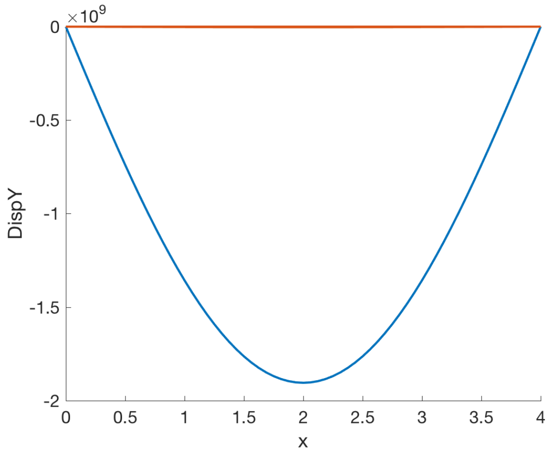
Figure 1: Beam

The cases we simulated in this project is shown in Table1.

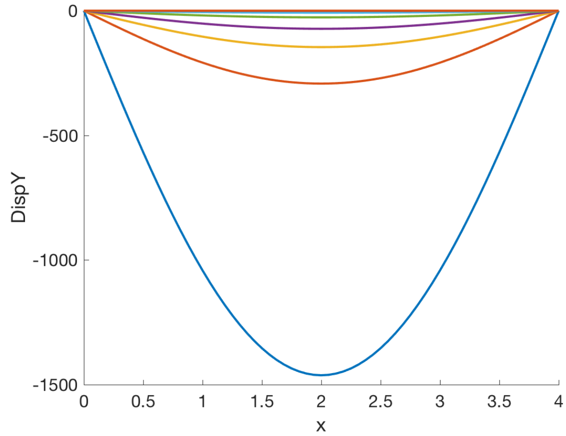
Case	a	a/L	Inertia=$\frac{a^4}{12}$
1	0.001	0.00025	8.33333E-14
2	0.005	0.00125	5.20833E-11
3	0.01	0.0025	8.33333E-10
4	0.02	0.005	1.33333E-08
5	0.05	0.0125	5.20833E-07
6	0.1	0.025	8.33333E-06
7	0.2	0.05	0.000133333
8	0.4	0.1	0.002133333

Table 1: Testing cases

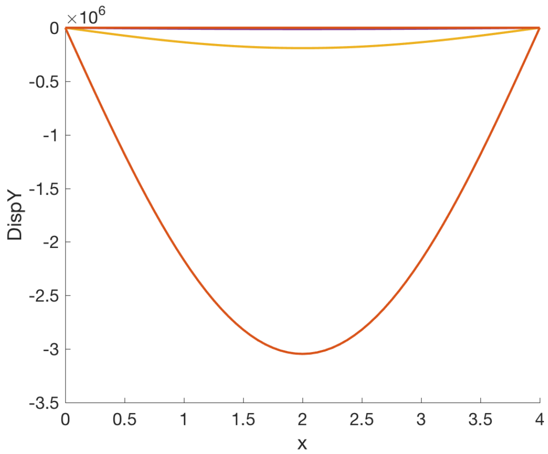
The convergence of displacement in the y direction with different slenderness ratio is shown in Figure.2. The value of a varies from 0.001 to 0.4. It can observed that the displacement calculated by Euler Bernoulli decreased rapidly when a/L increases.



(a) Euler Bernulli



(b) Timoshenko



(c) Reduced Timoshenko

Figure 2: Convergence of displacement

More detailed information of displacement using different beam theories is given in

case	Euler Bernoulli	Timoshenko	Reduced Timo-shenko
1	-1904840000	-1462.57	-1904200000
2	-3047620	-292.486	-3046430
3	-190476	-146.145	-190402
4	-11904.8	-72.6821	-11900.1
5	-304.762	-26.6898	-304.649
6	-19.0476	-8.2737	-19.043
7	-1.19048	-1.02486	-1.19144
8	-0.0744048	-0.073607	-0.07509

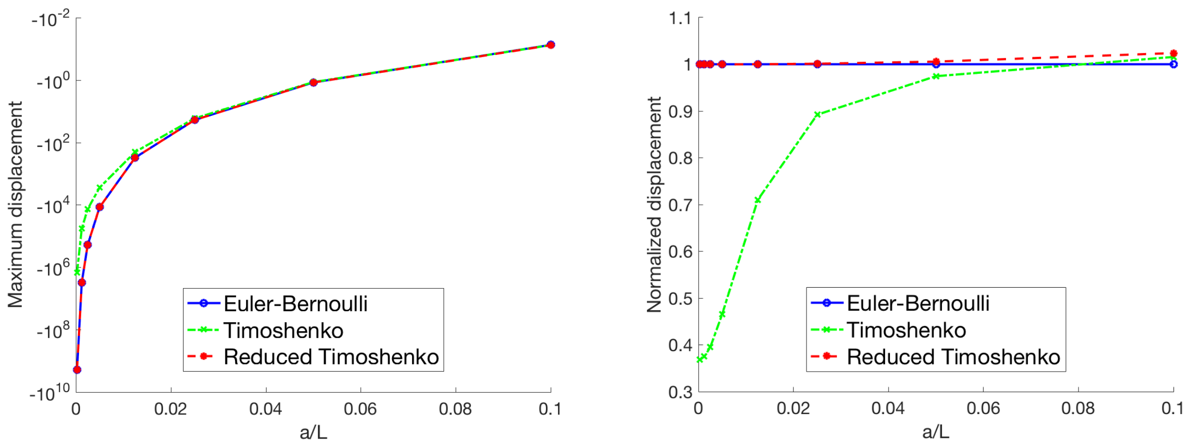
Table 2: Maximum displacement

To give a more vivid demonstration of the relationship between maximum displacement and a/L,

case	Euler Bernoulli	Timoshenko	Reduced Timoshenko
1	1.99991	1.54E-06	1.99913
2	1.99991	0.000191888	1.99902
3	1.99991	0.00153407	1.99902
4	1.99991	0.012207	1.99902
5	1.99991	0.175097	1.99902
6	1.99991	0.868354	1.99902
7	1.99991	1.7192	1.99902
8	1.99991	1.95916	1.99902

Table 3: Maximum moment

the above information is plotted in Figure.3. In Figure.3(a) for thick beams, the solutions of all the three methods are similar. Figure.3(b) can give a more vivid view of the different between these three methods, where the y axis is $exp(\frac{w-w_{Euler-Bernoulli}}{w_{Euler-Bernoulli}})$ The results of the Timoshenko theory are much more different from Euler Bernoulli when the cross section thickness decreases. This is due to the fact Timoshenko takes into account the shear deformation which is more realistic for thick beams. However, it can be observed when a/L is very small the result using Timoshenko theory is overly rigid which is called the shear locking effect. In slender beams, the effect of the shear deformation is negligible. This problem can be overcome by reduced integration in the Timoshenko method. Their results coincide with those obtained using the Euler-Bernoulli theory.



(a) Maximum displacement

(b) Normalized maximum rotation

Figure 3: Maximum displacements vs a/L

The detailed information of maximum moment is shown in Table3. It is plotted in Figure4. It demonstrates that the results obtained from the Euler Bernoulli theory and the Timoshenko theory with reduced integration are similar. Timoshenko theory with reduced integration is valid for both thick and slender beams. However Timoshenko theory with full integration fails for slender beams. Since the Euler Bernoulli theory assumes that the effect of transverse shear deformation is negligible for a slender beam, we cannot obtain shear stress from the Euler Bernoulli method. The shear forces are compared under Timoshenko theory with full and reduced integration. Figure.5 shows

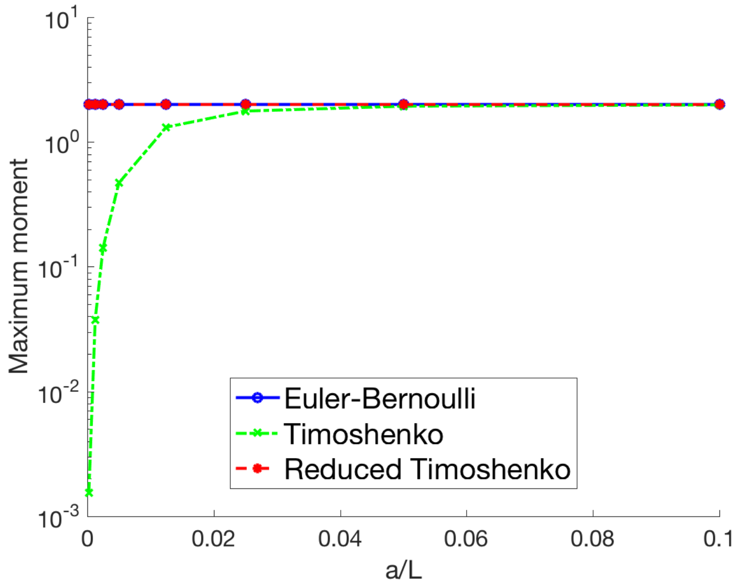


Figure 4: Maximum moment

that reduced Timoshenko theory can also accurately capture the transverse shear stress in the beam without compromising on the displacement.

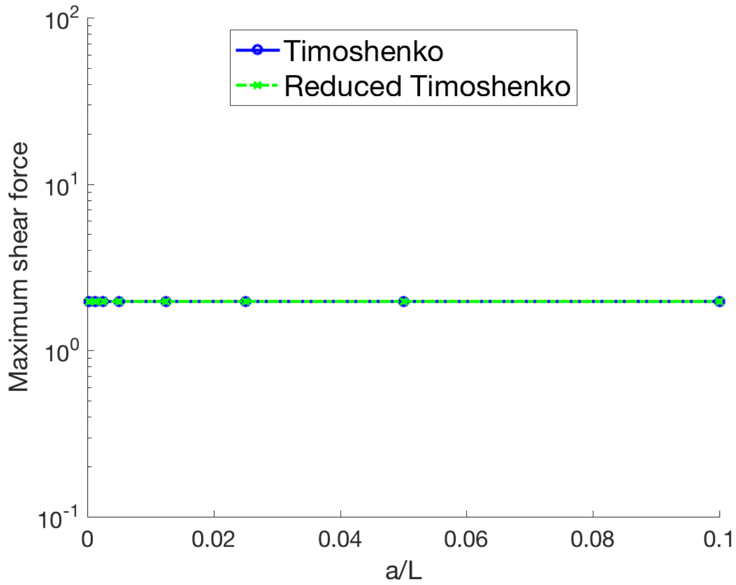


Figure 5: Maximum shear forces

2.1 Conclusions

In conclusion, for slender beams the Euler Bernoulli theory and Timoshenko theory with reduced integration are valid. For thick beams the Timoshenko beam element is more realistic than the Euler Bernoulli element as it includes transverse shear deformations.