Computational ~~Methods~~ Structural Mechanics and Dynamics    Aitor Bazán Escoda.
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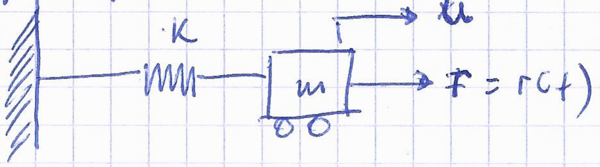
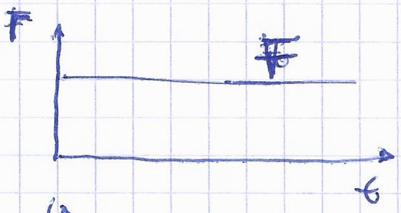
Assignment 9. On "Solid and structural Dynamics"

1) Effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system.




let $r(t) = F \rightarrow cte$




↳ Force applied suddenly ⇒ step function at $t=0$

the system depicted above can be modeled as:

$$m\ddot{u} + Ku = F \iff m\frac{d^2u}{dt^2} + Ku = F$$

The particular solution to the governing equation, since the force F has no time dependence, yields as:

$$u_p(t) = u_0$$

$$\dot{u}_p(t) = 0, \quad \ddot{u}_p(t) = 0 \Rightarrow Ku_0 = F \Rightarrow u_0 = \frac{F}{K} \Rightarrow u_p(t) = \frac{F}{K}$$

the homogeneous solution is thus:

$m\ddot{u} + Ku = 0 \Rightarrow$ we know, the solution for an ODE of this form is $u(t) = e^{pt} \Rightarrow mp^2e^{pt} + Ke^{pt} = 0$

$mp^2 + K = 0$ (it holds for any $t$)

$$p^2 = -\frac{K}{m}$$

$$p = \pm i\sqrt{\frac{K}{m}}$$

since $i = \sqrt{-1}$

and $\omega_n = \sqrt{\frac{K}{m}}$ → natural frequency

the solution yields:

$$u(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

and equivalently, $u(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t$

therefore, general equation reads as: $u(t) = u_h(t) + u_p(t)$

$$u(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t + \frac{F}{K}$$

$$u(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{F}{k}$$

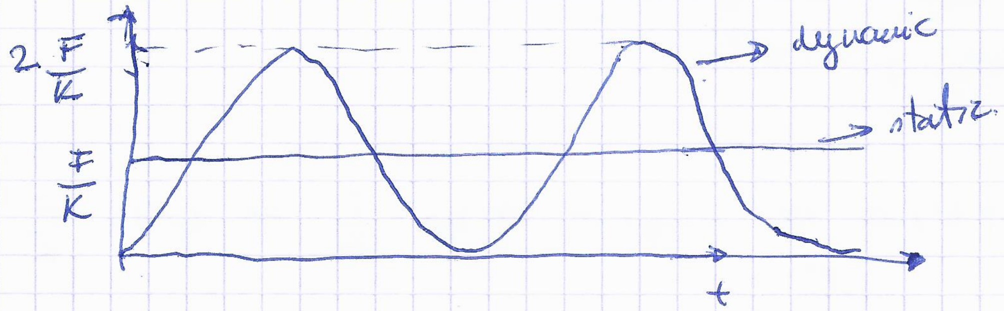
if we set as initial conditions:

$$t=0 \qquad u(t) = 0 \Rightarrow A_2 = -\frac{F}{K}$$
$$\dot{u}(t) = 0 \Rightarrow A_1 = 0$$

And, therefore, the final solution:

$$u(t) = \frac{F}{K}(1 - \cos \omega_n t) \quad \text{with} \quad \omega_n = \sqrt{\frac{K}{m}}$$

if we sketch the dynamic and static ~~sol~~ responses we have:




we see, the dynamic response is 2 times the static one.

• On the other side, what would happen if we apply a periodic force to the single-degree-of-freedom system?

In such case, $F(t) = F \sin \bar{\omega} t$ where $\bar{\omega}$ is the excitation frequency of the system.

the governing equation is now: $m\ddot{u} + Ku = F \sin \bar{\omega} t$

the homogeneous solution would be $\rightarrow$ $m\ddot{u} + Ku = 0$

$$u_h(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

the particular solution yields: $\rightarrow$ $u_p(t) = Y \sin \bar{\omega} t$

$$u_p(t) = Y \sin \bar{\omega} t$$
$$\dot{u}_p(t) = Y\bar{\omega} \cos \bar{\omega} t$$
$$\ddot{u}_p(t) = -Y\bar{\omega}^2 \sin \bar{\omega} t$$

$$-m\bar{\omega}^2 Y \sin(\bar{\omega} t) + KY \sin(\bar{\omega} t) = F \sin(\bar{\omega} t)$$
$$-m\bar{\omega}^2 Y + KY = F$$

$$Y = \frac{F}{K - m\bar{\omega}^2} = \frac{F/K}{1 - \frac{m\bar{\omega}^2}{K}} = \frac{F/k}{1 - \frac{\bar{\omega}^2}{K/m \rightarrow \omega_n^2}}$$

$$Y = \frac{F/k}{1 - \frac{\omega^2}{\omega_n^2}}$$

$$u(t) = u_n(t) + u_p(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{F_0/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} \sin \bar{\omega} t$$

$$\dot{u}(t) = A_1 \omega_n \cos \omega_n t - A_2 \omega_n \sin \omega_n t + \bar{\omega} \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} \cos \bar{\omega} t \quad \rightarrow \text{if } \eta^2 = \frac{\bar{\omega}^2}{\omega_n^2}$$

if we set as initial conditions:

$$u(t=0) = 0 \Rightarrow A_2 = 0$$

$$\dot{u}(t=0) = 0 \Rightarrow A_1 = -\eta \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} = -\frac{\bar{\omega}}{\omega_n} \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}}$$

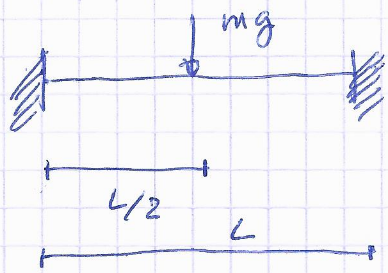
thus, the general equation yields:

$$u(t) = -\eta \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} \sin \omega_n t + \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} \sin \bar{\omega} t$$

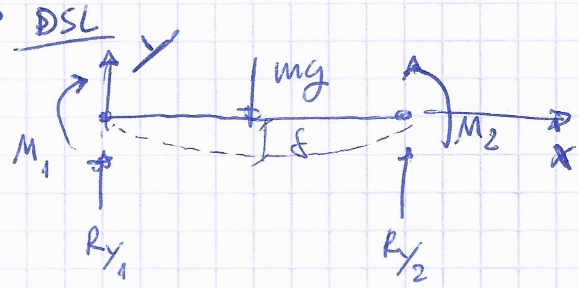
$$u(t) = \frac{F/k}{1 - \frac{\bar{\omega}^2}{\omega_n^2}} \left( \sin \bar{\omega} t - \frac{\bar{\omega}}{\omega_n} \sin \omega_n t \right)$$

- if $\omega_n = \bar{\omega} \rightarrow u(t) \rightarrow \infty$ → Resonance appears since the solution is unbounded
- if $\bar{\omega}$ is low → the response is quasi-static
- if $\bar{\omega}$ is high → response → 0

2)




→ DSL




• Estimate the natural frequency of vibration in terms of m, L, E and A.

First determine the effective K → $K = \frac{F}{\delta} \circ \frac{mg}{\delta}$

→ Neglect mass of the bar.

First, from equilibrium of forces and compatibility conditions, we know:

$$\Sigma F_Y = 0 \Rightarrow R_{Y_1} + R_{Y_2} - mg = 0$$

$$u(0) = u(L) = 0 \Rightarrow \frac{R_{Y_1} L}{EA} + \frac{R_{Y_2} L}{EA} = 0 \Rightarrow R_{Y_1} = -R_{Y_2} = \frac{mg}{2}$$

we can get the maximum displacement of the bar is in the middle of the same.

$$u\left(\frac{L}{2}\right) = \frac{R_{Y_1} L}{2EA} = \frac{mgL}{2EA}$$

$$K = \frac{mg}{u\left(\frac{L}{2}\right)} = \frac{mg\,2EA}{mgL} = \frac{2EA}{L}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2EA}{Lm}}$$

3) $m = \int_{\Omega^e} N^T N \rho \, dV$

For a linear Lagrange element with 2 nodes at $\xi = -1$ and $\xi = +1$, and being

$$N_i(\xi) = \prod_{j=1 (j \neq i)}^{n} \left( \frac{\xi - \xi_j}{\xi_i - \xi_j} \right)$$

thus, for 2D and integrating along the element,

$$N_1 = \frac{1}{2}(1-\xi) \qquad N_2 = \frac{1}{2}(1+\xi)$$

$$m = \rho \frac{A}{2} \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} |J| d\xi$$

$$= \frac{1}{8} AL\rho \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1+\xi)(1-\xi) & (1+\xi)^2 \end{bmatrix} d\xi = \frac{\rho AL}{8} \int_{-1}^{1} \begin{bmatrix} 1-2\xi+\xi^2 & 1-\xi^2 \\ 1-\xi^2 & 1+2\xi+\xi^2 \end{bmatrix} d\xi$$

$$= \frac{\rho AL}{8} \begin{bmatrix} \xi - \xi^2 + \frac{\xi^3}{3} & \xi - \frac{\xi^3}{3} \\ \xi - \frac{\xi^3}{3} & \xi + \xi^2 + \frac{\xi^3}{3} \end{bmatrix}_{-1}^{1} = \frac{\rho AL}{8} \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix} = \frac{\rho AL}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

4) obtain the mass matrix of a 2-noded linear element with a variable cross-sectional area that varies from $A_1$ to $A_2$.

Area variation can be expressed as:

$$A(\xi) = \sum_{i=1}^{2} N_i(\xi) A_i(\xi) = \frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi)$$

$$m = \rho L \int_{-1}^{1} N^T N A(\xi) d\xi = \frac{\rho L}{2} \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \left[ \frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi) \right] d\xi$$

$$= \frac{\rho L}{16} \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1+\xi)(1-\xi) & (1+\xi)^2 \end{bmatrix} \left[ A_1(1-\xi) + A_2(1+\xi) \right] d\xi$$

$$= \frac{\rho L}{16} \int_{-1}^{1} \left[ A_1 \begin{bmatrix} (1-\xi)^3 & (1-\xi)^2(1+\xi) \\ (1+\xi)(1-\xi)^2 & (1+\xi)^2(1-\xi) \end{bmatrix} + A_2 \begin{bmatrix} (1-\xi)^2(1+\xi) & (1-\xi)(1+\xi)^2 \\ (1+\xi)^2(1-\xi) & (1+\xi)^3 \end{bmatrix} \right] d\xi$$

$$= \frac{\rho L}{16} \left[ A_1 \begin{bmatrix} \xi - \frac{3\xi^2}{2} + \xi^3 - \frac{\xi^4}{4} & \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} \\ \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \end{bmatrix} + \right.$$

$$\left. + A_2 \begin{bmatrix} \xi - \frac{\xi^2}{2} + \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \\ \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} & \xi + \frac{3\xi^2}{2} + \frac{\xi^3}{3} + \frac{\xi^4}{4} \end{bmatrix} \right]_{-1}^{1}$$

$$= \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix}$$

5) A uniform 2-noded bar element is allowed to move in 3D space. The nodes have only translational d.o.f. Diagonal mass matrix of the element?

$$\bar{M}_L^e = m = \frac{\rho L A}{2} I_6 = \begin{bmatrix} \frac{\rho LA}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\rho LA}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho LA}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho LA}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho LA}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\rho LA}{2} \end{bmatrix}$$

* All translational masses must be retained in the local mass matrix. If we think in Newton's 2nd law for an element moving in a translational rigid body motion $u_R$ with acceleration $\ddot{u}_R$, $f_R = M^e \ddot{u}_R$, where $M^e$ is the translational mass. Therefore it cannot be zero.