UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER IN COMPUTATION MECHANICS AND NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 9

by

Renan Alessio

Barcelona, May of 2020

Index

1- Introduction

The goal of the assignment is to employ the concepts of Solid and Structural Dynamics.

2 – Point 1

The differential equation which needs to be solved to find the effects of F on the displacement u(t) and natural frequency of the system in slide 6 [2]:

$$\mathrm{m\ddot{u}} + \mathrm{ku} = \mathrm{F} \quad (1)$$

Equation 1 has total solution u which can be split between a particular solution u_p and a homogeneous solution u_h. Therefore, u can be defined as a linear combination of both solutions:

$$u = u_p + u_h \quad (2)$$

The homogeneous solution for the type of differential equation in Equation 1 is:

$$u_h = C_1 sen\omega t + C_2 cos\omega t \quad (3)$$

Considering the particular solution as [5]:

$$u_p = A \quad (4)$$

Replacing Equation 4 in Equation 1, A and the particular solution are defined as:

$$u_p = A = \frac{F}{k} \quad (5)$$

Rewriting Equation 2:

$$u = C_1 sen\omega t + C_2 cos\omega t + \frac{F}{k} \quad (6)$$

Considering the following initial conditions:

$$u(0) = u_0, \quad \dot{u}(t) = v_0$$

Replacing the initial conditions in Equation 6 and solving for the constants C_1 and C_2:

$$C_1 = \frac{V_0}{\omega}, \quad C_2 = u_0 - \frac{F}{k}$$

Replacing the values of the constants C_1 and C_2 in Equation 6, the total solution of Equation 1 is:

$$u = \frac{V_0}{\omega} sen\omega t + \left(u_0 - \frac{F}{k}\right) cos\omega t + \frac{F}{k} \quad (7)$$

The natural frequency of the system can be defined as:

$$\omega = \sqrt{\frac{k}{m}} \tag{8}$$

Therefore, the force F applied to the system increases the value of the displacement for each time t and has no effect on the natural frequency of the system. Such observations are expected, since F will influence on the displacement response of the system but, since the natural frequency is a feature intrinsic to the system only, it has no effect on the system's natural frequency.

3 – Point 2

Considering the bar, with neglectable mass itself, and weight attached to its middle point, depicted in Figure 1:

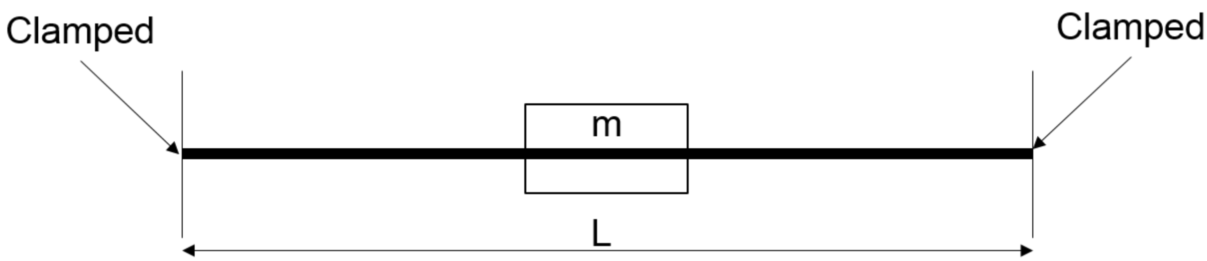

Figure 1. Structure considered for Point 2.

The natural frequency can be calculated as follows [2]:

$$\omega = \sqrt{\frac{K_{effective}}{m}} \tag{9}$$

The $K_{effective}$ in suce case is defined as [3]:

$$K_{effective} = \frac{192EI}{L^3} \tag{10}$$

Particularizing Equation 10 to a rod with square cross section:

$$K_{effective} = \frac{16EA^2}{L^3} \tag{11}$$

Replacing Equation 11 in Equation 9:

$$\omega = \sqrt{\frac{16EA^2}{mL^3}} \tag{12}$$

4 – Point 3

Considering the following definition of the mass matrix **m** [2]:

$$\boldsymbol{m} = \int \boldsymbol{N}^T \boldsymbol{N} \rho dV \tag{13}$$

Where the matrix with shape functions **N** is defined as [4]:

$$\boldsymbol{N} = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \tag{14}$$

We can calculate each component of the mass matrix **m** as follows:

$$m_{\boldsymbol{11}} = \frac{\rho A}{L^2} \int (L^2 - 2xL + x^2) dx = \frac{\rho AL}{3} \tag{15}$$

$$m_{\boldsymbol{22}} = \frac{\rho A}{L^2} \int (L^2 - 2xL + x^2 dx = \frac{\rho AL}{3} \tag{16}$$

$$m_{\boldsymbol{12}} = m_{\boldsymbol{21}} = \frac{\rho A}{L^2} \int (Lx - x^2) dx = \frac{\rho AL}{6} \tag{17}$$

Rewriting the mass matrix **m**:

$$\boldsymbol{m} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix} \tag{18}$$

Equation 18 is mass matrix shown in slide 17 [2].

5 – Point 4

Considering a linear displacement element with length L and varying cross-section area from A_1 to A_2, the mass matrix **m** can be defined as:

$$\boldsymbol{m} = \int \boldsymbol{N}^T \boldsymbol{N} \rho A(x) dx \tag{19}$$

Where A(x) is defined as:

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{A_2 - A_1}{L} x + A_1 \tag{120}$$

Computing each component of the mass matrix **m** separately:

$$m_{\boldsymbol{11}} = \frac{A_2 - A_1}{L} \int (L^2 x - 2x^2 L + x^3) dx + A1 \int (L^2 - 2xL + x^2 dx) = \frac{\rho L(A_2 + 3A_1)}{12} \tag{21}$$

$$m_{\mathbf{22}} = \frac{A_2 - A_1}{L}\int (L^2 x - 2x^2 L + x^3)dx + A1\int (L^2 - 2xL + x^2 dx) = \frac{\rho L(A_2 + 3A_1)}{12} \tag{22}$$

$$m_{\mathbf{12}} = m_{\mathbf{21}} = \frac{A_2 - A_1}{L}\int (Lx^2 - x^3)dx + A1\int (L^2 - 2xL + x^2 dx) = \frac{\rho L(A_2 + A_1)}{12} \tag{23}$$

Rewriting the mass matrix **m**:

$$\boldsymbol{m} = \begin{bmatrix} \frac{\rho L(A_2 + 3A_1)}{12} & \frac{\rho L(A_2 + A_1)}{12} \\ \frac{\rho L(A_2 + A_1)}{12} & \frac{\rho L(A_2 + 3A_1)}{12} \end{bmatrix} \tag{24}$$

6 – Point 5

The diagonal mass matrix of a bar element in a 3D space is defined as [2]:

$$\boldsymbol{m} = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ Symm. & & & & & 1 \end{bmatrix} \tag{25}$$

7 – Discussion and Conclusions

The current assignment enabled the employment of concepts regarding the topic of Solid and Structural Dynamics. The concept of mass matrix was applied in depth in the current assignment. The mass matrix has interesting variations – a lumped form or a consistent form. The lumped form, or diagonal form, is particularly interesting from the computational cost point of view. Since it is a diagonal matrix, the operations involving such entity are reduced. Also, the diagonal matrix can be obtained applying a Finite Element approach or a Finite Difference approach. Nevertheless, the drawback of a lumped mass matrix is the discontinuous acceleration field inside an element, besides reducing the accuracy of the solution. The consistent mass matrix on the other hand provides a solution with a linear acceleration field inside the element. The accuracy is also improved in comparison with the lumped mass matrix, but the computational cost is greater.

8 – References

[1] – Assignment on Solid and Structural Dynamics, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[2] – Presentation "Solid & Structural Dynamics", Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[3] – Miguel, L. F. F., "Introdução ao Estudo de Vibrações", Vibrations I, Federal University of Rio Grande do Sul, 2006.

[4] – Oñate, E., "Structural Analysis with the Finite Element Method – Linear Statics", Vol.1, Springer, 2009.

[5] – Zill, D.G., Cullen, M. R., "Differential Equations", Vol.1, Pearson, 2001.