Computational Structural Mechanics and Dynamics

As9 Dynamics

Ye Mao

mao.ye@estudiant.upc.edu

Master of Numerical methods on engineering - Universitat Politècnica de Catalunya

Dynamics

1. **In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?**

[Answer]

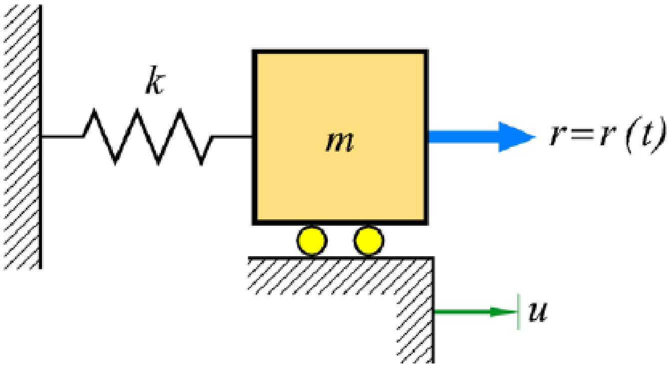

Fig.1 Model

We can depict it in Figure 1, and obtain the following equation

$$m\ddot{u} + ku = r$$

$$m\frac{d^2u}{dt^2} + ku = r$$

We assume two case to investigate the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system.

Case 1, F is constant:

Assume $m = 1\ k = 1\ r = F = 0\ \ u_0 = 0$

Case 2, F is natural frequency:

Assume $m = 1\ k = 1\ r(t) = F = \sin(\omega t)\ \ u_0 = 0$

Case 3, F is constant:

Assume $m = 1\ k = 4\ r = F = 0\ \ u_0 = 0$

Case 4, F is natural frequency:

Assume $m = 1\ k = 4\ r(t) = F = \sin(\omega t)\ \ u_0 = 0$

Natural frequency $f = \frac{\omega}{2\pi}$

The period is T= $f = \frac{1}{f} = \frac{2\pi}{\omega}$

$$u = \bar{u}sin\sin(\omega t)$$

Where $\bar{u}$ is amplitude of motion and $\omega = \sqrt{\frac{k}{m}}$ is the natural frequency of vibration.

Then, we obtain the force equation as:

$$\frac{d^2u}{dt^2} + u = 0, in\ case\ 1\ \omega = 1$$

$$\frac{d^2u}{dt^2} + u = \sin(t)\,, in\ case\ 2\ \omega = 1$$

$$\frac{d^2u}{dt^2} + 4u = 0, in\ case\ 3\ \omega = 2$$

$$\frac{d^2u}{dt^2} + 4u = \sin(2t)\,, in\ case\ 4\ \omega = 2$$

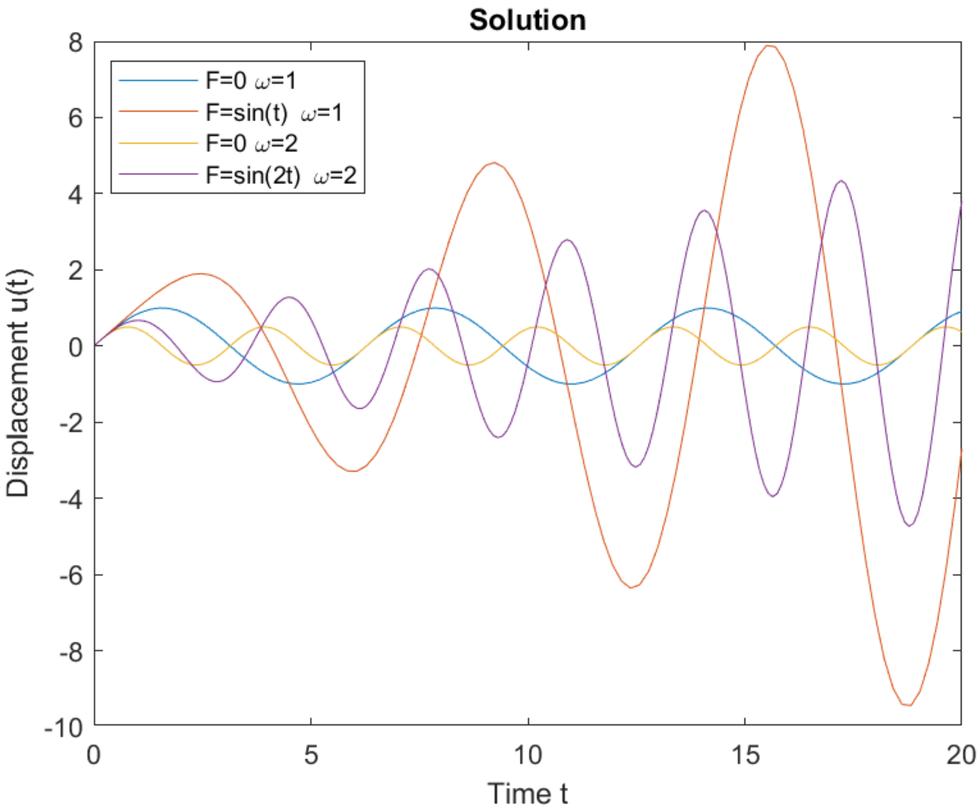

Fig.2 Solution

In conclusion, the constant force could keep the amplitude of motion while nature frequency force will lead to the amplitude of motion expanded. At the same time, the amplitude of motion will decrease while ω is increasing (the same as frequency increasing).

2. **A weight whose mass is m is placed at the middle of uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the naturel frequency of vibration in terms of $m, L, E\ and\ A$. Suggestion: First determine the effective k.**

[Answer]

From this model, the Deflection in the middle point of the bar is

$$u_{\frac{L}{2}} = \frac{\frac{1}{2}mgl}{EA}$$

Stiffness K is:

$$K = \frac{mg}{u_{\frac{L}{2}}}$$

Then, Naturel frequency of vibration is

$$f_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{mg}{mu_{\frac{L}{2}}}} = \sqrt{\frac{mgEA}{m\frac{1}{2}mgl}} = \sqrt{\frac{2EA}{ml}}$$

3. **Use the expression on slide 18 to derive the mass matrix of slide 17.**
[Answer]
The consistent element mass matrix is calculated as:

$$m = \int N^T N \rho dV = \int N^T N \rho A |J| d\xi$$

Using the isoparametric representation of the two-node bar element:

$$m = \rho A \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} |J| d\xi$$

$$= \frac{1}{4}\rho A \frac{1}{2} L \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} d\xi$$

$$= \rho A \frac{1}{8} L \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} d\xi$$

$$= \rho A \frac{1}{8} L \int_{-1}^{1} \begin{bmatrix} 1-2\xi+\xi^2 & 1-\xi^2 \\ 1-\xi^2 & 1-2\xi+\xi^2 \end{bmatrix} d\xi$$

$$= \rho A \frac{1}{8} L \begin{bmatrix} \xi-\xi^2+\frac{\xi^3}{3} & \xi-\frac{\xi^3}{3} \\ \xi-\frac{\xi^3}{3} & \xi-\xi^2+\frac{\xi^3}{3} \end{bmatrix}_{-1}^{1}$$

$$= \rho A \frac{1}{8} L \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} \frac{\rho A L}{3} & \frac{\rho A L}{6} \\ \frac{\rho A L}{6} & \frac{\rho A L}{3} \end{bmatrix}$$

4. **Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2.**
[Answer]

$$A(\xi) = \sum_{i=1}^{2} N_i(\xi) A_i = \frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi)$$

$$m = \int N^T N \rho dV = \int N^T N \rho A |J| d\xi$$

$$= \frac{1}{4}\rho \frac{1}{2} L \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} (\frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi)) d\xi$$

$$= \frac{1}{16}\rho L \int_{-1}^{1} (A_1 \begin{bmatrix} (1-\xi)^3 & (1-\xi)^2(1+\xi) \\ (1-\xi)^2(1+\xi) & (1+\xi)^2(1-\xi) \end{bmatrix}$$

$$+ A_2 \begin{bmatrix} (1-\xi)^2(1+\xi) & (1-\xi)(1+\xi)^2 \\ (1-\xi)(1+\xi)^2 & (1+\xi)^3 \end{bmatrix}) d\xi$$

$$= \frac{1}{16}\rho L \int_{-1}^{1} (A_1 \begin{bmatrix} 1 - 3\xi + 3\xi^2 - \xi^3 & 1 - \xi + \xi^2 + \xi^3 \\ 1 - \xi + \xi^2 + \xi^3 & 1 + \xi - \xi^2 - \xi^3 \end{bmatrix} + A_2 \begin{bmatrix} 1 - \xi - \xi^2 + \xi^3 & 1 + \xi - \xi^2 - \xi^3 \\ 1 + \xi - \xi^2 - \xi^3 & 1 + 3\xi + 3\xi^2 + \xi^3 \end{bmatrix}) d\xi$$

$$= \frac{1}{16}\rho L (A_1 \begin{bmatrix} \xi - \frac{3\xi^2}{2} + \xi^3 - \frac{\xi^4}{4} & \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} \\ \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \end{bmatrix}_{-1}^{1} + A_2 \begin{bmatrix} \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \\ \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} & \xi + \frac{3\xi^2}{2} + \xi^3 + \frac{\xi^4}{4} \end{bmatrix}_{-1}^{1}) d\xi$$

$$= \begin{bmatrix} \frac{\rho L(3A_1 + A_2)}{12} & \frac{\rho L(A_1 + A_2)}{12} \\ \frac{\rho L(A_1 + A_2)}{12} & \frac{\rho L(A_1 + 3A_2)}{12} \end{bmatrix}$$

5. **A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass of the element?**

   [Answer]

$$m = \frac{\rho A L}{2} I_6 = \frac{\rho A L}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$