CSMD: Assignment 5

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Assignment 5.1

1

The coordinates of the nodes of the triangle are:

$$r_1 = 0 \quad r_2 = r_3 = a \quad z_1 = z_2 = 0 \quad z_3 = b$$

The shape functions using triangular coordinates are:

$$N_1 = \zeta_1 \; N_2 = \zeta_2 \; N_3 = \zeta_3$$

The Jacobian matrix is:

$$\boldsymbol{J} = \begin{bmatrix} 1 & 1 & 1 \\ \sum r_i \frac{\partial N_i}{\partial \theta_1} & \sum r_i \frac{\partial N_i}{\partial \theta_2} & \sum r_i \frac{\partial N_i}{\partial \theta_3} \\ \sum z_i \frac{\partial N_i}{\partial \theta_1} & \sum z_i \frac{\partial N_i}{\partial \theta_2} & \sum z_i \frac{\partial N_i}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{bmatrix}$$

$$J = \frac{det\boldsymbol{J}}{2} = \frac{ab}{2}$$

The shape functions derivatives with respect to r and z coordinates are:

$$\begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial r} & \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial r} & \frac{\partial N_3}{\partial z} \end{bmatrix} = \frac{1}{2J} \begin{bmatrix} z_2 - z_3 & r_3 - r_2 \\ z_3 - z_1 & r_1 - r_3 \\ z_1 - z_2 & r_2 - r_1 \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} -b & 0 \\ b & -a \\ 0 & a \end{bmatrix}$$

The strain-displacement matrix will be:

$$\boldsymbol{B_i} = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix}$$

$$\boldsymbol{B} = \frac{1}{ab} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab\zeta_1}{r} & 0 & \frac{ab\zeta_2}{r} & 0 & \frac{ab\zeta_3}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

The stiffness matrix will be:

$$K^e = 2\pi \int_A r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}dA$$

Changing to triangular variables:

$$K^e = 2\pi \int r\left(\zeta_1, \zeta_2, \zeta_3\right)\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}Jd\zeta_1 d\zeta_2 d\zeta_3$$

Where:

$$r\left(\zeta_1,\zeta_2,\zeta_3\right)=\zeta_1 r_1+\zeta_2 r_2+\zeta_3 r_3=\left(\zeta_2+\zeta_3\right)a$$

$$\boldsymbol{EB}=\frac{E}{ab}\begin{bmatrix}-b & 0 & b & 0 & 0 & 0\\ 0 & 0 & 0 & -a & 0 & a\\ \frac{ab\zeta_1}{r} & 0 & \frac{ab\zeta_2}{r} & 0 & \frac{ab\zeta_3}{r} & 0\\ 0 & \frac{-b}{2} & \frac{-a}{2} & \frac{b}{2} & \frac{a}{2} & 0\end{bmatrix}$$

$$\boldsymbol{B}^T\boldsymbol{EB}=\frac{E}{(ab)^2}\begin{bmatrix}b^2+\left(\frac{ab\zeta 1}{r}\right)^2 & 0 & -b^2+\left(\frac{ab}{r}\right)^2\zeta_1\zeta_2 & 0 & \left(\frac{ab}{r}\right)^2\zeta_1\zeta_3 & 0\\ 0 & \frac{b^2}{2} & \frac{ba}{2} & -\frac{b^2}{2} & -\frac{ba}{2} & 0\\ -b^2+\left(\frac{ab}{r}\right)^2\zeta_1\zeta_2 & \frac{ba}{2} & b^2+\left(\frac{ab\zeta 2}{r}\right)^2+\frac{a^2}{2} & -\frac{ba}{2} & \left(\frac{ab}{r}\right)^2\zeta_2\zeta_3-\frac{a^2}{2} & 0\\ 0 & -\frac{b^2}{2} & -\frac{ba}{2} & a^2+\frac{b^2}{2} & \frac{ba}{2} & -a^2\\ \left(\frac{ab}{r}\right)^2\zeta_1\zeta_3 & -\frac{ba}{2} & \left(\frac{ab}{r}\right)^2\zeta_2\zeta_3-\frac{a^2}{2} & \frac{ba}{2} & \left(\frac{ab\zeta_3}{r}\right)^2+\frac{a^2}{2} & 0\\ 0 & 0 & 0 & -a^2 & 0 & a^2\end{bmatrix}$$

Using 1 Gauss point ($\zeta_1=\zeta_2=\zeta_3=\frac{1}{3}$, w = 1):

$$K^e=2\pi\frac{ab}{2}\int r\left(\zeta_1,\zeta_2,\zeta_3\right)\boldsymbol{B}^T\boldsymbol{EB}d\zeta_1 d\zeta_2 d\zeta_3=2\pi\frac{ab}{2}\frac{2a}{3}\boldsymbol{B}^T\boldsymbol{EB}$$

Thus:

$$K^e=2\pi E\frac{1}{3b}\begin{bmatrix}b^2+\frac{b^2}{4} & 0 & -b^2+\frac{b^2}{4} & 0 & \frac{b^2}{4} & 0\\ 0 & \frac{b^2}{2} & \frac{ba}{2} & -\frac{b^2}{2} & -\frac{ba}{2} & 0\\ -b^2+\frac{b^2}{4} & \frac{ba}{2} & b^2+\frac{b^2}{4}+\frac{a^2}{2} & -\frac{ba}{2} & \frac{b^2}{4}-\frac{a^2}{2} & 0\\ 0 & -\frac{b^2}{2} & -\frac{ba}{2} & a^2+\frac{b^2}{2} & \frac{ba}{2} & -a^2\\ \frac{b^2}{4} & -\frac{ba}{2} & \frac{b^2}{4}-\frac{a^2}{2} & \frac{ba}{2} & \frac{b^2}{4}+\frac{a^2}{2} & 0\\ 0 & 0 & 0 & -a^2 & 0 & a^2\end{bmatrix}$$

2

The sum of columns 2,4 and 6 is 0:

$$\begin{bmatrix}0\\ \frac{b^2}{2}\\ \frac{ba}{2}\\ -\frac{b^2}{2}\\ -\frac{ba}{2}\\ 0\end{bmatrix}+\begin{bmatrix}0\\ -\frac{b^2}{2}\\ -\frac{ba}{2}\\ a^2+\frac{b^2}{2}\\ \frac{ba}{2}\\ -a^2\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ -a^2\\ 0\\ a^2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

The sum of columns 1,3 and 5 is:

$$\begin{bmatrix} b^2+\frac{b^2}{4} \\ 0 \\ -b^2+\frac{b^2}{4} \\ 0 \\ \frac{b^2}{4} \\ 0 \end{bmatrix} + \begin{bmatrix} -b^2+\frac{b^2}{4} \\ \frac{ba}{2} \\ b^2+\frac{b^2}{4}+\frac{a^2}{2} \\ -\frac{ba}{2} \\ \frac{b^2}{4}-\frac{a^2}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{b^2}{4} \\ -\frac{ba}{2} \\ \frac{b^2}{4}-\frac{a^2}{2} \\ \frac{ba}{2} \\ \frac{b^2}{4}+\frac{a^2}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3b^2}{4} \\ 0 \\ \frac{3b^2}{4} \\ 0 \\ \frac{3b^2}{4} \\ 0 \end{bmatrix}$$

The sum of columns (and rows) 2,4 and 6 must be 0 to account for the fact that rigid body motions can appear in the z direction if it is not constrained : columns 2,4 and 6 multiply the vertical displacements of the three nodes of the triangle; if their sum is zero and rigid body motions are applied ($u_{z1} = u_{z2} = u_{z3}$) then the vertical forces obtained will be 0 and there will be no stresses in the element due to the rigid body motion.

This analysis cannot be extended to columns (and rows) 1, 3 and 5 because the displacements in the radial direction will be always prescribed for r=0 (the axis of symmetry is not deforming). Thus, a motion like $u_{r1} = u_{r2} = u_{r3}$ will produce force and stresses in the radial direction. Moreover, since the element is not a 2D element but a ring element, any displacement in the radial direction will produce a deformation and stresses will appear.

3

The elemental force vector due to gravity forces is:

$$\int_{V^e} \boldsymbol{N}^T \boldsymbol{b} dV = 2\pi \int_{A^e} r \boldsymbol{N}^T \rho \boldsymbol{g} dA$$

In triangular coordinates:

$$2\pi \int_{A^e} r\boldsymbol{N}^T \rho \boldsymbol{g} dA = 2\pi \int_{A^e} \rho r\left(\zeta_1,\zeta_2,\zeta_3\right) \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} JdA = 2\pi \int_{A^e} \rho r\left(\zeta_1,\zeta_2,\zeta_3\right) \begin{bmatrix} 0 \\ -g\zeta_1 \\ 0 \\ -g\zeta_2 \\ 0 \\ -g\zeta_3 \end{bmatrix} JdA$$

Where:

$$r\left(\zeta_1,\zeta_2,\zeta_3\right) = \zeta_1 r_1 + \zeta_2 r_2 + \zeta_3 r_3 = \left(\zeta_2+\zeta_3\right) a$$

Using one integration point $\zeta_1 = \zeta_2 = \zeta_3 = \frac{1}{3}$ and assuming constant density:

$$2\pi \int_{A^e} \rho r \boldsymbol{N}^T \boldsymbol{g} JdA = 2\pi\rho \frac{2a}{3}\frac{ab}{2} \begin{bmatrix} 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \end{bmatrix} = 2\pi\rho\frac{a^2 b}{3} \begin{bmatrix} 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \end{bmatrix}$$

Assignment 5.2

The force vector due to a uniform pressure p acting normal to the face 1-2-3 ($\zeta_4 = 0$) can be computed as:

$$\boldsymbol{f}_p = \int \boldsymbol{N}^t \boldsymbol{t} dA = \int \boldsymbol{N}^T (\zeta_1, \zeta_2, \zeta_3, 0)(-p)\,\hat{\boldsymbol{n}} dA$$

Where :

$$\boldsymbol{N}(\zeta_1, \zeta_2, \zeta_3, 0) = \begin{bmatrix} \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_1 & 0 & 0 & \zeta_2 & 0 & 0 & \zeta_3 & 0 & 0 & 0 \end{bmatrix}$$

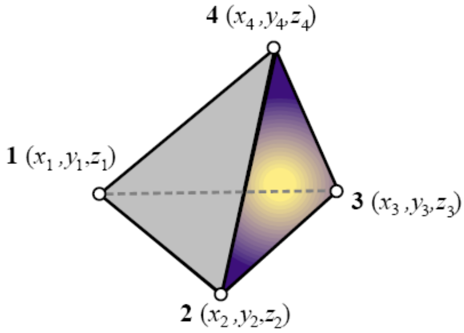

Figure 1: Linear tetrahedron.

The outward normal to the plane can be expressed as:

$$\hat{\boldsymbol{n}} = \frac{(\boldsymbol{x_3} - \boldsymbol{x_1}) \times (\boldsymbol{x_2} - \boldsymbol{x_1})}{|(\boldsymbol{x_3} - \boldsymbol{x_1}) \times (\boldsymbol{x_2} - \boldsymbol{x_1})|} = \frac{(\boldsymbol{x_3} - \boldsymbol{x_1}) \times (\boldsymbol{x_2} - \boldsymbol{x_1})}{2A} =$$

$$= \frac{1}{2A} \begin{bmatrix} (y_3 - y_1)(z_2 - z_1) - (z_3 - z_1)(y_2 - y_1) \\ (z_3 - z_1)(x_2 - x_1) - (x_3 - x_1)(z_2 - z_1) \\ (x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1) \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Where A is the area of the face. The force due to the pressure is thus:

$$\boldsymbol{f}_p = \frac{1}{2A} \int \boldsymbol{N}^T (\zeta_1, \zeta_2, \zeta_3, \zeta_4 = 0)(-p) \begin{bmatrix} a \\ b \\ c \end{bmatrix} dA$$

Last expression can be integrated in the centroid of the face ($\zeta_1 = \zeta_2 = \zeta_3 = \frac{1}{3}, \zeta_4 = 0$, w = 1):

$$\boldsymbol{f}_p = \frac{1}{2} \boldsymbol{N}^T \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right)(-p) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{p}{6} \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \\ a \\ b \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If p and ρ were not uniform they should have been taken into account when integrating.