Homework 1

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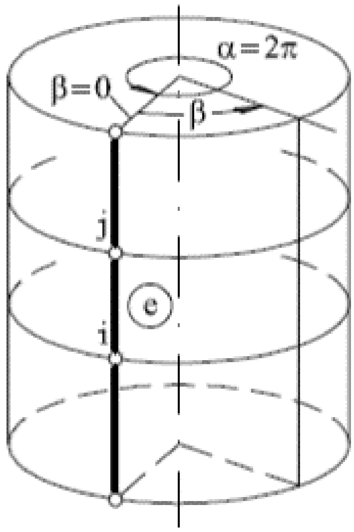
Part1 :

The local displacements are expanded in Fouries series along the circular (circumferential) direction. 1D finite elements are used to discretize the transversal (meridional) direction. The "length" of the structure is a whole circumference for the axi-symmetric shells and, therefore, angle α is replaced by 2π.It can be seen from the figure 1.



The local displacement vector is written as

$\mathbf{u}^{'}(s,\beta) = \sum_{l=1}^{m}\sum_{i=1}^{n} \mathbf{S}^{l}(\beta)\mathbf{N}_i(s)\mathbf{a}_i^{'l}$

$u^{'}(s,\beta) = [u_0^{'}, v_0^{'}, w_0^{'}, \theta_s, \theta_t]^T$; $\mathbf{a}^{'l} = [u_{0i}^{'l}, v_{0i}^{'l}, w_{0i}^{'l}, \theta_{si}^{l}, \theta_{ti}^{l}]^T$;;

$\mathbf{S}^l(\beta)$ is defined as : $\begin{bmatrix} S^l & 0 & 0 & 0 & 0 \\ 0 & C^l & 0 & 0 & 0 \\ 0 & 0 & S^l & 0 & 0 \\ 0 & 0 & 0 & S^l & 0 \\ 0 & 0 & 0 & 0 & C^l \end{bmatrix}$ Where $S^l = S^l(\beta) = \sin\gamma\beta; C^l = C^l(\beta) = \cos\gamma\beta$

$\mathbf{N}_i(s) = N_i(s)\mathbf{I}_5$;

For axisymmetric shells $\alpha = 2\pi, so\ \gamma = \frac{\pi L}{\alpha} = L/2$;

So, $S^l = S^l(\beta) = \sin\frac{L\beta}{2}; C^l = C^l(\beta) = \cos\frac{L\beta}{2}$

$\theta_s\ and\ \theta_t$ are the rotations of the normal vector contained in the planes sn and st, respectively. The discretization process leads to the following relationship between the local generalized strains and the nodal modal displacement amplitudes: $\hat{\epsilon}^{'}(s,\beta) = \sum_{l=1}^{m}\sum_{i=1}^{n} \mathbf{S}^{l}(\beta)\mathbf{B}_i^{'l}(s)\mathbf{a}_i^{'l}$

$$\hat{\boldsymbol{\varepsilon}}' = \begin{Bmatrix} \hat{\boldsymbol{\varepsilon}}'_m \\ \hat{\boldsymbol{\varepsilon}}'_b \\ \hat{\boldsymbol{\varepsilon}}'_s \end{Bmatrix}; \begin{cases} \hat{\boldsymbol{\varepsilon}}'_m = \begin{Bmatrix} \frac{\partial u'_0}{\partial s} \\ \frac{1}{r}\frac{\partial v_0}{\partial \beta} + \frac{u'_0}{r}C - \frac{w'_0}{r}S \\ \frac{\partial v'_0}{\partial s} - \frac{1}{r}\frac{\partial u'_0}{\partial \beta} - \frac{v'_0}{r}C \end{Bmatrix} \\ \hat{\boldsymbol{\varepsilon}}'_b = \begin{Bmatrix} \frac{\partial \theta_s}{\partial s} \\ \frac{1}{r}\frac{\partial \theta_t}{\partial \beta} + \frac{\theta_s}{r}C \\ \frac{\partial \theta_t}{\partial s} + \frac{1}{r}\frac{\partial \theta_s}{\partial \beta} - \frac{\theta_t}{r}C \end{Bmatrix} \\ \hat{\boldsymbol{\varepsilon}}'_s = \begin{Bmatrix} \frac{\partial w'_0}{\partial s} - \theta_s \\ \frac{1}{r}\frac{\partial w'_0}{\partial \beta} + \frac{v'_0}{r}S - \theta_t \end{Bmatrix} \end{cases}$$

$$\mathbf{B}_i'^l = \begin{Bmatrix} \mathbf{B}_{m_i}'^l \\ \mathbf{B}_{b_i}'^l \\ \mathbf{B}_{s_i}'^l \end{Bmatrix}; \begin{cases} \mathbf{B}_{m_i}'^l = \begin{bmatrix} \frac{\partial N_i}{\partial s} & 0 & 0 & 0\,0 \\ \frac{N_i}{r}C & -\frac{N_i}{r}\gamma & -\frac{N_i}{r}S & 0\,0 \\ \frac{N_i}{r}\gamma & (\frac{\partial N_i}{\partial s} - \frac{N_i}{r}C) & 0 & 0\,0 \end{bmatrix} \\ \mathbf{B}_{b_i}'^l = \begin{bmatrix} 0\,0\,0 & \frac{\partial N_i}{\partial s} & 0 \\ 0\,0\,0 & \frac{N_i}{r}C & -\frac{N_i}{r}\gamma \\ 0\,0\,0 & \frac{N_i}{r}\gamma & \left[\frac{\partial N_i}{\partial s} - \frac{N_i}{r}C\right] \end{bmatrix} \\ \mathbf{B}_{s_i}'^l = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial s} & -N_i & 0 \\ 0 & \frac{N_i}{r}S & \frac{N_i}{r}\gamma & 0 & -N_i \end{bmatrix} \end{cases}$$

$S = \sin\phi^{(e)}, C = \cos\phi^{(e)};$

Following the standard discretization procedure, the uncoupled system of stiffness equations is obtained. The local stiffness matrix and the equivalent nodal modal force amplitude vector for distributed loading for a troncoconical strip are:

$$[\mathbf{K}_{ij}'^{ll}]^{(e)} = \frac{\alpha}{2}\int_{a^{(e)}} [\mathbf{B}_i'^l]^T \hat{\mathbf{D}}' \mathbf{B}_j'^l r ds$$

$$[\mathbf{f}_i^l]^{(e)} = \frac{\alpha}{2}\int_{a^{(e)}} N_i \mathbf{t}^l r ds$$

The loads are expanded similarly as for the displacements.Nodal point loads are directly assembled in the global equivalent nodal modal force vector in the standard manner. The nodal modal amplitude vectors for distributed forces $\mathbf{t}^l$ and point loads $\mathbf{p}^l$ are computed by: $[\mathbf{f}_i^{l^e}] = \frac{\alpha(=2\pi)}{2}\int_{\mathbf{a}^e} \mathbf{N}_i \mathbf{t}^l ds = \pi \int_{\mathbf{a}^e} \mathbf{N}_i \mathbf{t}^l ds$ where $\mathbf{t}^l$ is the amplitude vector for a distributed load for the l-th harmonic term given by:

$\mathbf{t}^l = \frac{2}{\pi l}[f_s(C_0^l - C_1^l), f_\beta(S_1^l - S_0^l), f_z(C_0^l - C_1^l), m_s(C_0^l - C_1^l), m_\beta(S_1^l - S_0^l), 0]^T$

For a point load acting at a node with global number i we have:
$f_i^l = p_i^l = \frac{2}{\alpha(=2\pi)}[F_{s_i}, S_i^l, F_{\beta_i}, C_i^l, F_{zi}, S_i^l, M_{xi}, S_i^l, M_{\beta i}, C_i^l, 0]^T$; and
$C_i^l = \cos\gamma\beta_i; S_i^l = \sin\gamma\beta_i$

Part 2:
Most analytical solutions for axisymmetric shells are based on Kirchhoff assumption for the orthogonality of the normal rotation. This hypothesis, though only acceptable for thin shell situations, can be applied to many problems of practical interest and analytical solutions are available for cylindrical reservoirs, spherical and conical domes, circular plates etc. The key difference between Kirchhoff and Reissner-Mindlin theories is the assumption made for the rotation of the normal. Kirchhoff theory establishes that, as the thickness is small, the normals to the generatrix remain straight and orthogonal to the generatrix after deformation. Hence, the normal rotation coincides with the slope of the generatrix at each point.In mathematical form we can write

$\theta = \frac{\partial w'}{\partial s}\Big|_{z'=0}$
Substituting the expression for $\frac{\partial w'}{\partial s}$: $\frac{\partial w'}{\partial s} = \frac{\partial w_0'}{\partial s} + \frac{u_0'}{R_s} - z'\frac{\theta}{R_s}$ gives $\theta = \frac{\partial w_0'}{\partial s} + \frac{u_0'}{R_s}$

The equation for $\gamma_{x'z'}$ can be written as: $\gamma_{x'z'} = \frac{1}{C_s}(\frac{\partial w_0'}{\partial s} + \frac{u_0'}{R_s} - \theta) = 0$
i.e. the Kirchhoff orthogonality condition is equivalent to neglecting the effect of transverse shear deformation, as expected.For thin shells; $C_s = C_\alpha = 1$. The local displacement vector is now defined as: $u' = \begin{bmatrix} u_0', & w_0', & \frac{\partial w_0'}{\partial s}\end{bmatrix}^T$
The expressions for the axial and circumferential strains are deduced to:
$\epsilon_x' = \frac{1}{C_s}[\frac{\partial u_0'}{\partial s} - \frac{w_0'}{R_s} - z'(\frac{\partial^2 w_0'}{\partial s^2} + \frac{\partial}{\partial s}(\frac{u_0'}{R_s}))$

$\epsilon_y' = \frac{1}{C_\alpha}[\frac{u_0'\cos\phi - w_0'\sin\phi}{r} - z'\frac{\cos\phi}{r}(\frac{\partial w_0'}{\partial s} + \frac{u_0'}{R_s})]$

The generalized strain vector is: $\hat{\epsilon}' = \mathbf{S}_2\begin{bmatrix}\hat{\epsilon}_m' \\ \hat{\epsilon}_b'\end{bmatrix}$; where $\mathbf{S}_2 = \begin{bmatrix}\mathbf{S} & -z'\mathbf{S}\end{bmatrix}$ and $\mathbf{S}$ is defined as $\mathbf{S} = \begin{bmatrix}\frac{1}{C_s} & 0 \\ 0 & \frac{1}{C_\alpha}\end{bmatrix}$ and the membrane and bending generalized strains are: $\epsilon_m' = \begin{bmatrix}\frac{\partial u_0'}{\partial s} - \frac{w_0'}{R_s} \\ \frac{u_0'\cos\phi - w_0'\sin\phi}{r}\end{bmatrix}$ and $\epsilon_b' = \begin{bmatrix}\frac{\partial^2 w_0'}{\partial s^2} + \frac{\partial}{\partial s}(\frac{u_0'}{R_s}) \\ \frac{\cos\phi}{r}(\frac{\partial w_0'}{\partial s} + \frac{u_0'}{R_s})\end{bmatrix}$;

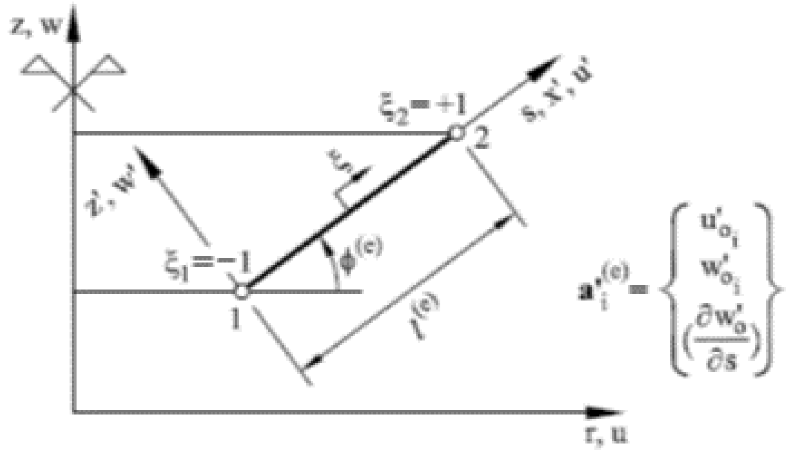
Considering $R_s = \infty$, the generalized strain vectors simplify to: $\epsilon_m' = \begin{bmatrix}\frac{\partial u_0'}{\partial s} \\ \frac{u_0'\cos\phi - w_0'\sin\phi}{r}\end{bmatrix}$ and $\epsilon_b' = \begin{bmatrix}\frac{\partial^2 w_0'}{\partial s^2} \\ \frac{\cos\phi}{r}(\frac{\partial w_0'}{\partial s})\end{bmatrix}$;

A C^1continuous interpolation must be used for the normal displacement w_0' to satisfy element conformity. A simpler C^0 continuous Lagrange approximation can however be employed for the tangential displacement u_0'. Also as the element is straight $\frac{\partial w_0'}{\partial s} = \frac{\partial w_0'}{\partial x'}$ and $\frac{\partial^2 w_0'}{\partial s^2} = \frac{\partial^2 w_0'}{\partial x'^2}$

The simplest element based on Kirchhoff theory has two nodes. The tangential displacement is linearly interpolated as $u_0' = \sum_{i=1}^2 N_i^u u_{0i}' with \;\; N_i^u = \frac{1+\xi\xi_i}{2}$;

The followingC_1 continuous approximation is chosen for $w_0^{'} = \sum_{i=1}^{2}[{N_i}^w w_{0i}^{'} + \overline{N_i}^w \frac{\partial w_0^{'}}{\partial s}_i]$ where ${N_i}^w$ and $\overline{N_i}^w$ are the cubic 1D Hermite shape functions. The local generalized strain matrix is written as: $\widehat{\epsilon}' = [\mathbf{B}_1^{'}, \mathbf{B}_2^{'}]\mathbf{a}'^{(e)} = \mathbf{B}^{'}\mathbf{a}'^{(e)}$ with

$$B_i^{'} = \begin{bmatrix} B_{m_i}^{'} \\ \cdots \\ B_{b_i}^{'} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i^u}{\partial s} & 0 & 0 \\ \frac{N_i^u \cos\phi}{r} & -\frac{N_i^w \cos\phi}{r} & \frac{\overline{N_i}^w \sin\phi}{r} \\ 0 & \frac{\partial^2 N_i^w}{\partial s^2} & \frac{\partial^2 \overline{N_i}^w}{\partial s^2} \\ 0 & \frac{\cos\phi}{r}\frac{\partial N_i^w}{\partial s} & \frac{\cos\phi}{r}\frac{\partial \overline{N_i}^w}{\partial s} \end{bmatrix}$$



The explicit form of the $B^{'} = \begin{bmatrix} -\frac{1}{L^e} & 0 & 0 & \frac{1}{L^e} & 0 & 0 \\ \frac{1-\xi}{2x}C^e & (-N_1^w)\frac{S^e}{4r} & (-\overline{N_1}^w)\frac{S^e}{4r} & \frac{1+\xi}{2r}C^e & (-N_2^w)\frac{S^e}{4r} & (-\overline{N_2}^w)\frac{S^e}{4r} \\ 0 & \frac{6\xi}{L^{e2}} & \frac{2(3\xi-1)}{L^{e2}} & 0 & \frac{-6\xi}{L^{e2}} & \frac{-2(1+3\xi)}{L^{e2}} \\ 0 & \frac{(\xi^2-1)3C^e}{2rL^e} & \frac{H_1 C^e}{2rL^e} & 0 & \frac{(1-\xi^2)3C^e}{2rL^e} & \frac{H_2 C^e}{2rL^e} \end{bmatrix}$

Where $C^e = \cos\phi^e; S^(e) = \sin\phi^e;$
$N_i^w = \frac{2+3\xi\xi_i - \xi^3\xi_i}{4}$
$\overline{N_i}^w = \frac{\xi^3+\xi^2\xi_i-\xi-\xi_i}{4}; H_i = 3\xi^2 + 2\xi\xi_i - 1$

Integration Rules:

A two-point quadrature is recommended for computing the integrals containing rational terms.Good results are obtained however with the simplest reduced one-point quadrature.This is equivalent to making $\xi = 0$ *and* $r = r_m in$ $\mathbf{B}^{'}$. A more accurate expression for the stiffness matrix using a seven-point quadrature can also be found.