Computational Structural Mechanics and Dynamics
Assignment 1

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1 Problem 1

1.1 a

For element 1, the angle is $90+\alpha$ and the length is $\frac{L}{c}$, leading to the following k-matrix.

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{bmatrix}$$

rewriting the matrix in the expanded form leads to

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & -sc^2 & -cs^2 & sc^2 & 0 & 0 & 0 & 0 \\ -sc^2 & c^3 & sc^2 & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For element 2, the angle is 90 with length L, leading to the following k-matrix.

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

which is written i the expanded form as:

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For element 3, the angle is $90-\alpha$ with length $\frac{L}{c}$, leading to the following k-matrix.

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -cs^2 & -sc^2 & cs^2 & sc^2 \\ -sc^2 & -c^3 & sc^2 & c^3 \end{bmatrix}$$

which is written in the expanded form as:

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & sc^2 & 0 & 0 & 0 & 0 & -cs^2 & -sc^2 \\ sc^2 & c^3 & 0 & 0 & 0 & 0 & -sc^2 & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -sc^2 & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix}$$

Doing the assembling process (compatibility and equilibrium) will lead the the following system:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & -sc^2 \\ 0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -sc^2 & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix}\begin{bmatrix} u1_x \\ u1_y \\ u2_x \\ u2_y \\ u3_x \\ u3_y \\ u4_x \\ u4_y \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The 5th row and column is related to the x movement of node 3. They are all zeros because truss structure only have stiffness in the axial direction, therefore there is no stiffness to the movement of node 3 in the x-direction from the structure itself.

1.2 b

Applying the boundry conditions to th equations will lead to:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & -sc^2 \\ 0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -sc^2 & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{bmatrix}\begin{bmatrix} u1_x \\ u1_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be reduced the 2*2 system as:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u1_x \\ u1_y \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

1.3 c

solving for the displacements:

$$u_1x = \frac{HL}{2EAcs^2}, \qquad u_1y = \frac{-PL}{EA(1+2c^3)}$$

when α goes to zero, the solution becomes

$$u_1x = \infty, \qquad u_1y = \frac{-PL}{3EA}$$

This solution makes sense because as α goes to zero, the structure will reduce to a just 3 parallel beams, which will have no resistance to the movement of node 1 in the x- direction, that's why the solution blows.

when α goes to $\frac{\pi}{2}$, the solution becomes

$$u_1x = \infty, \qquad u_1y = \frac{-PL}{EA}$$

still the displacement in the x-direction goes to infinity, since the structure has no resistance in that direction. Furthermore, the new structure has only 1 element in the y resisting the movement, therefore, the y-displacement is 3 times larger.

1.4 d

Calculating the axial force for each element, through calculating the local displacement, from which we get the elongation and then the force

For element 1:

$$u'1x = -s\frac{HL}{2EAcs^2} + c\frac{-PL}{EA(1+2c^3)}$$
$$u'2x = 0$$
$$d_1 = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$
$$F_1 = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}$$

For element 2:

$$u'1x = \frac{-PL}{EA(1+2c^3)}$$
$$u'2x = 0$$
$$d_2 = \frac{PL}{EA(1+2c^3)}$$
$$F_2 = \frac{P}{(1+2c^3)}$$

For element 3:

$$u'1x = s\frac{HL}{2EAcs^2} + c\frac{-PL}{EA(1+2c^3)}$$
$$u'2x = 0$$
$$d_3 = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$
$$F_3 = -\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}$$

when α goes to zero, F_1 and F_3 will tend to infinity(the solution explodes). That is due to the fact that, there is no resistance to the displacement in the x-direction, and thus the elongation of the 2 elements will go to infinity and thus the forces. This doesn't happen is element 2, because the axial force only depends in the y-displacement which is fine.

2 Question 2

Adding a node to the second element so the K-matrix will be 10*10

by the same procedure we can achieve the global K-matrix as follows

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & 0 & -cs^2 & cs^2 & 0 & 0 & 0 & 0 & -cs^2 & -sc^2 \\ & 2+2c^3 & sc^2 & -c^3 & 0 & -2 & 0 & 0 & -sc^2 & -c^3 \\ & & cs^2 & -sc^2 & & 0 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & symm & & & 4 & 0 & -2 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & 2 & 0 & 0 \\ & & & & & & & & cs^2 & sc^2 \\ & & & & & & & & & c^3 \end{bmatrix}$$

adding the boundary conditions and the reduced k-matrix will be

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} cs^2 & 0 & 0 & 0 \\ & 2+2c^3 & 0 & -2 \\ & symm & 0 & 0 \\ & & & 4 \end{bmatrix}$$

the determinant of the reduced k-matrix is zero, which means the matrix is singular and the solution can't be obtained. That is due to adding extra degrees of freedom with no resistance which makes the system has multiple solutions (which is not there in reality), which can't be solved. Therefore, adding extra degrees of freedom is not always better to maintain an achievable solution.