Computational Structural Mechanics and Dynamics 27.02.17

* Assignment 3.1

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad -(1) \qquad \mu = G = \frac{E}{2(1+\nu)} \quad -(2)$$

1. Relation for E, ν in terms of λ and μ

From equation (2)
we get,

$$\mu = \frac{E}{2(1+\nu)} \rightarrow 2\mu = \frac{E}{(1+\nu)}$$ – substitute in (1)

we get,

$$\lambda = \frac{2\mu\nu}{(1-2\nu)} \qquad \therefore \lambda(1-2\nu) = 2\mu\nu$$

$$\lambda - 2\nu\lambda = 2\mu\nu$$

$$\lambda = \nu(2\lambda + 2\mu)$$

$$\boxed{\therefore \nu = \frac{\lambda}{2(\lambda+\mu)}}$$ – substitute in (2) to obtain E

$$E = 2\mu(1+\nu)$$

$$= 2\mu\left(1 + \frac{\lambda}{2\lambda+2\mu}\right) = 2\mu\left(\frac{3\lambda+2\mu}{2\lambda+2\mu}\right)$$

$$\boxed{E = \mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)}$$

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② Elastic matrix for plane stress and plane strain

Ⓐ PLANE STRESS :→

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{Bmatrix} \quad \text{(A)}$$

Term ①

$$\frac{E}{1-\nu^2} = \frac{\mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)}{1-\left(\frac{\lambda}{(2\lambda+2\mu)}\right)^2}$$

$$= \frac{\mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)}{\frac{4\lambda^2+8\mu\lambda+4\mu^2-\lambda^2}{4(\lambda+\mu)^2}}$$

$$= \frac{\mu\left(\frac{\cancel{3\lambda+2\mu}}{\cancel{\lambda+\mu}}\right)}{\frac{\cancel{(3\lambda+2\mu)}(\lambda+2\mu)}{4(\lambda+\mu)^{\cancel{2}}}}$$

$$\boxed{\frac{E}{1-\nu^2} = \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}} \quad — \quad (3)$$

Term ②

$$\frac{1-\nu}{2} = \frac{1-\left(\frac{\lambda}{2\lambda+2\mu}\right)}{2}$$

$$\frac{1-\nu}{2} = \frac{2\lambda+2\mu-\lambda}{\frac{2}{2}(\lambda+\mu)} = \frac{(\lambda+2\mu)}{(\lambda+\mu)} \quad (4)$$

Substitute ③, ④ and ν in equation Ⓐ, we get

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \begin{bmatrix} 1 & \frac{\lambda}{(2\lambda+2\mu)} & 0 \\ \frac{\lambda}{2\lambda+2\mu} & 01 & 0 \\ 0 & 0 & \frac{(\lambda+2\mu)}{(\lambda+\mu)} \end{bmatrix} [\bar{e}]$$

$$\boxed{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} & \frac{2\mu\lambda}{(\lambda+2\mu)} & 0 \\ \frac{2\mu\lambda}{(\lambda+2\mu)} & \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} & 0 \\ 0 & 0 & 4\mu \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix}}$$

Plane stress in terms of λ and μ

(B) PLANE STRAIN :- ▶

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} \quad \text{— (B).}$$

$\boxed{\text{Term (1)}}$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\cancel{\mu}\left(\frac{\cancel{3\lambda+2\mu}}{\cancel{\lambda+\mu}}\right)\left(\frac{\lambda+2\mu}{2(\cancel{\lambda+\mu})}\right)}{\left(\frac{\cancel{3\lambda+2\mu}}{\cancel{2\lambda+2\mu}}\right)\left(\frac{\cancel{\mu}}{\cancel{\lambda+\mu}}\right)}$$

$$\boxed{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \lambda + 2\mu} \quad \text{— (5).}$$

$\boxed{\text{Term (2)}}$

$$\frac{\nu}{1-\nu} = \frac{\frac{\lambda}{\cancel{(2\mu+2\lambda)}}}{\frac{\lambda+2\mu}{\cancel{(2\mu+2\lambda)}}} = \boxed{\frac{\lambda}{2\mu+\lambda}} \quad \text{— (6)}$$

$\boxed{\text{Term (3)}}$

$$\frac{1-2\nu}{2(1-\nu)} = \frac{\frac{\mu}{\cancel{(\lambda+\mu)}}}{\frac{\cancel{2}(\lambda+2\mu)}{\cancel{2}(\cancel{\lambda+\mu})}} = \boxed{\frac{\mu}{\lambda+2\mu}} \quad \text{— (7)}$$

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Substituting ⑤, ⑥, ⑦ in Ⓑ, we get

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = (\lambda+2\mu) \begin{bmatrix} 1 & \frac{\lambda}{(\lambda+2\mu)} & 0 \\ \frac{\lambda}{\lambda+2\mu} & 1 & 0 \\ 0 & 0 & \frac{\mu}{\lambda+2\mu} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}}$$

Plane strain in terms of λ and μ.

From the above relation we get matrix E for plane strain as

$$E = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

③ Splitting stress-strain matrix 'E' for plane strain as E_λ and E_μ

$$E = \underbrace{\begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_\lambda} + \underbrace{\begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}_{E_\mu}$$

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4. $\boxed{E_\lambda \text{ and } E_\mu \text{ in terms of } E \text{ and } \nu}$

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

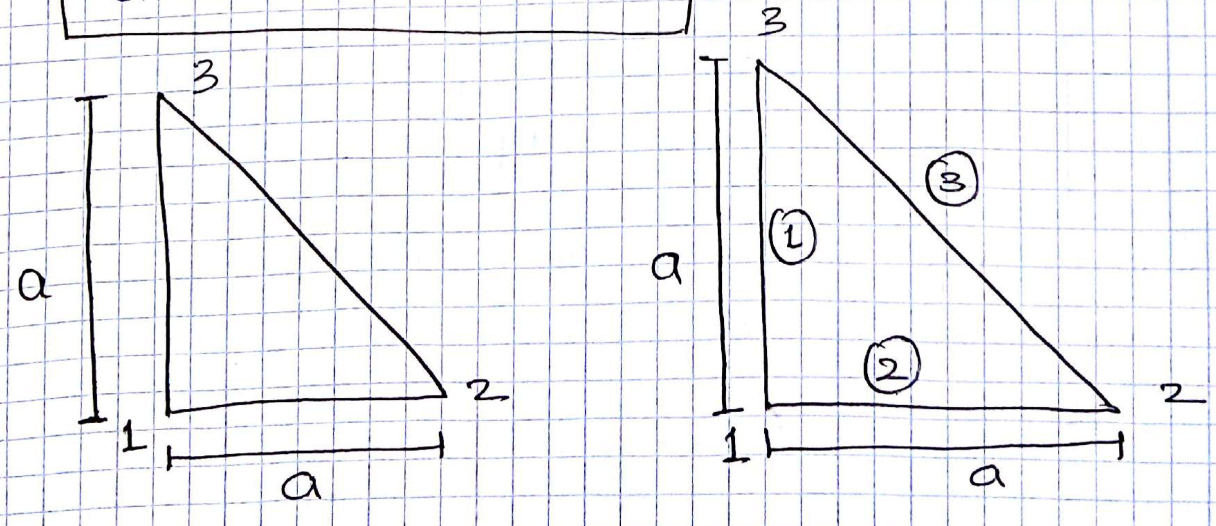
* **ASSIGNMENT 3.2.**

Given Data:→

$a=1$, $b=1$ material parameters → E, ν

initially $\boxed{\nu = 0}$

1. $\boxed{\text{Calculate } K_{tri} \text{ and } K_{bar} \text{ for discrete models}}$



for bar elements $A_1 = A_2 = A_3$

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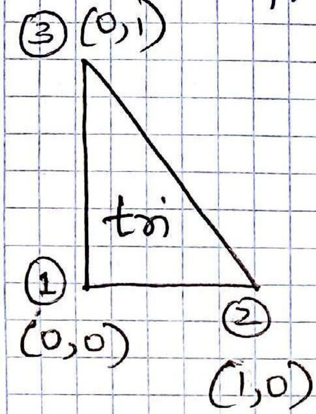
$$K^{e}_{tri} = \int_{\Omega_e} h\, B^T E B \, d\Omega$$

for linear triangle with plane stress:

$$K_{tri} = \frac{h}{4A^2} \underbrace{\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}}_{B^T} \begin{bmatrix} \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{B} \end{bmatrix}$$



$$A = \frac{1}{2} \times 1 \times 1 = \frac{1}{2} \; ; \; h = 1$$

$$K_{tri} = \frac{1}{4\left(\frac{1}{2}\right)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{B} \end{bmatrix}$$

$$\tilde{E} = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \text{ when } \nu = 0$$

we get

$$\boxed{K_{tri} = \frac{E}{2} \begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1.0 \end{bmatrix}}$$

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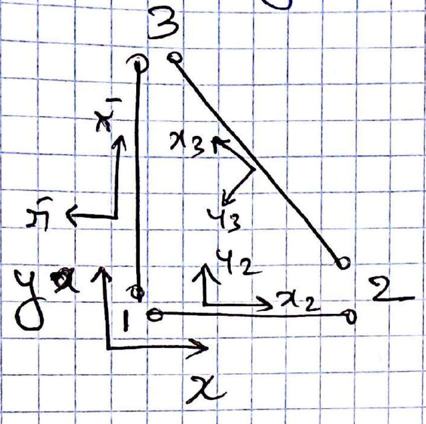
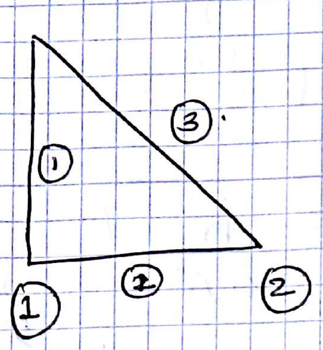
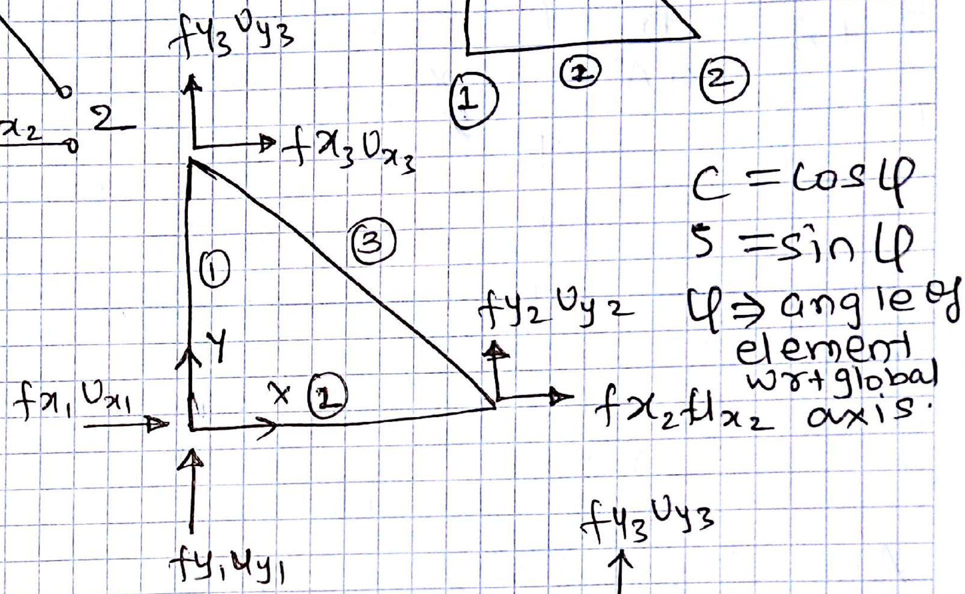
Computing stiffness for bar elements







$C = \cos\varphi$

$S = \sin\varphi$

$\varphi \Rightarrow$ angle of element wrt global axis.

$L_1 = L_2 = L$

$L_3 = \sqrt{2}\,L$

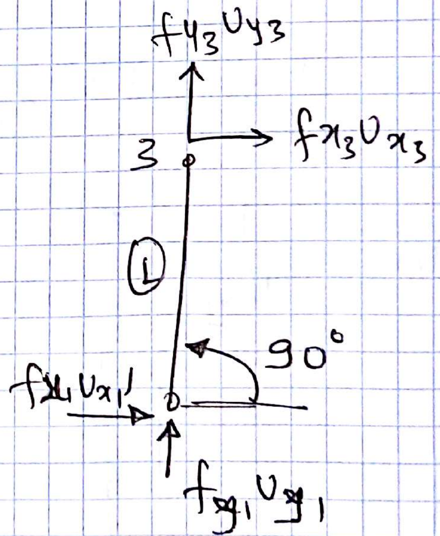
$A_1 = A_2 = A_3$

$E_1 = E_2 = E_3$

for Element 1

$\boxed{\varphi = 90^\circ}$

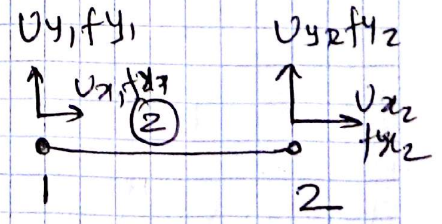
$$K^1 = \frac{EA_1}{L} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$



similarly for $\boxed{\text{element 2}}$

$\boxed{\varphi = 0^\circ}$

$$K^2 = \frac{EA_2}{L} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$



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for element ③ $\varphi = 135°$ and

$L^3 = L\sqrt{2}$

$$K^3 = \frac{EA_3}{L\sqrt{2}} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6. \end{array}$$

Assembling the elemental stiffness to get the global stiffness matrix. we get:

$$K_{bar} = \frac{E}{2} \begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & A_1 + \frac{0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} & \frac{0.5A_3}{\sqrt{2}} \\ 0 & 0 & \frac{-0.5A_3}{\sqrt{2}} & \frac{0.5A_3}{\sqrt{2}} & \frac{0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} \\ 0 & 0 & \frac{-0.5A_3}{\sqrt{2}} & \frac{0.5A_3}{\sqrt{2}} & \frac{0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} \\ 0 & -A_1 & \frac{0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} & \frac{-0.5A_3}{\sqrt{2}} & A_1 + \frac{0.5A_3}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ v_y^2 \\ v_x^3 \\ v_y^3 \end{pmatrix}$$

② / ③ $\boxed{\text{Equivalence of } K_{bar} \text{ and } K_{tri}}$

$\boxed{K_{tri} \neq K_{bar}}$ for any value of Areas as the two problems are completely diff-erent one is a plate and the other is a truss.

• $K_{tri} \neq K_{Bar}$ for any of the values since the stiffness and displacements corresponding to Uy_1, Uy_2, Ux_3 and Uy_3 will not contribute to stiffness of truss for fx_1 force whereas it will affect the stiffness in case of triangular plane stress case.

Q.4 $\boxed{\text{Considering } \nu \neq 0}$

When $\nu \neq 0$ the stiffness matrix will be as below.

$$K_{tri} = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{3-\nu}{2} & \frac{\nu+1}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{\nu+1}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

• the poissons ratio will affect the whole stiffness matrix. It gives us a ratio between Lateral and longitudnal strain, ie. Lateral contraction due to applied stress.

When $\nu = \frac{1}{2}$ material is incompressible $\Rightarrow$ Volume Constant

$\nu = 0$ means no lateral Contraction.

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