

Universitat Politècnica de Catalunya

Computational Structural Mechanics and Dynamics - Assignment 1 : Direct Stiffness Method

Submitted by

Shaikh Salik Ahmad Firdos Ahmad
MSc Computational Mechanics
2015-17

Submitted to

Dr Miguel Cervera
Professor
UPC-BarcelonaTech

February 13 2017

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1 Direct Stiffness Method

Consider the truss problem defined in the Figure. All geometric and material properties: L , α , E and A , as well as the applied forces P and H , are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

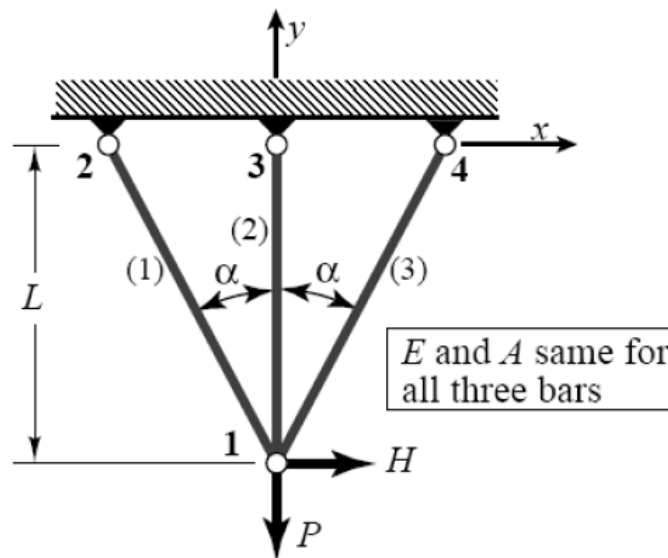


Figure 1.1: Truss Problem to be solved

1.1 Explanation for row 5 and column 5

The element 2 or the bar 2 is a vertical member of the given truss structure, So its contributions to the global stiffness matrix are only in the y -direction. The contributions in the horizontal components of stiffness by this member are thus all 0.

1.2 Computation of Master Stiffness Matrix:

In order to compute the master stiffness matrix we need to first compute the elemental stiffness matrices. To set this problem we will first define the number of unknowns for this truss, here we have 4 nodes with each node having a DOF of 2. The total DOF for the system is thus 8. We can see that from the figure Below, We can see that f_{xi} and f_{yi} are the nodal forces and u_{xi} and u_{yi} are nodal displacements with respect to the Global Coordinate system, here ($i = 1, 2, 3, 4$) where i corresponds to the node number.

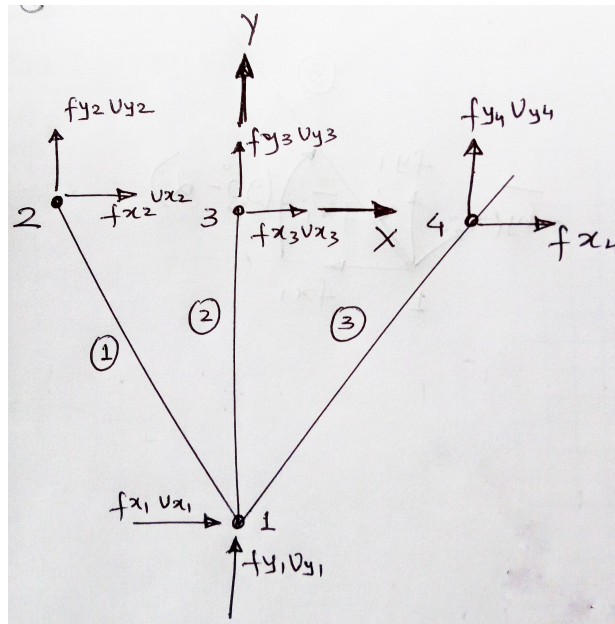


Figure 1.2: Truss - Elements(3), Nodes(4), DOF(8), Nodal Forces(f_{xi}, f_{yi}), Nodal Displacements (u_{xi}, u_{yi})

We will use the following notations, $\cos \alpha = c$ and $\sin \alpha = s$ In this Truss, $L^{(2)} = L$ and by using trigonometry from figure(1.1) we can see that $L^{(1)} = L^{(3)} = \frac{L}{c}$. The Young's Modulus E and the Cross sectional area A for all the trusses is the same.

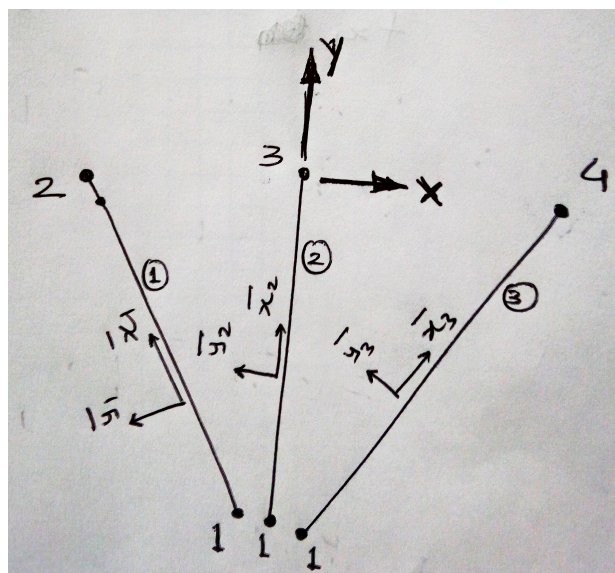


Figure 1.3: Local and global coordinate system for all elements of the truss

Now we shall compute the elemental stiffness matrices with respect to the Global Coordinate system. So we need to transform all the locally existing force and Nodal Displacements to Global ones.

Global Displacement transformation

$$\bar{u}^e = T^e u^e$$

where \bar{u}^e is the displacement vector w.r.t Local coordinate system and u^e vector is w.r.t global coordinate system.

Global force transformation

$$f^e = (T^e)^T \bar{f}^e$$

where \bar{f}^e is the force vector w.r.t Local coordinate system and f^e vector is w.r.t global coordinate system.

Globalisation of Stiffness Matrix

$$K^e = (T^e)^T \bar{K}^e T^e$$

where \bar{K}^e is the elemental stiffness matrix w.r.t Local coordinate system and K^e matrix is w.r.t global coordinate system.

Here the matrix T^e can be defined as below:

$$T^e = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

and eventually K^e can be written as below:

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

here $c = \cos\phi$ and $s = \sin\phi$ where ϕ is the angle between the global coordinate system and the local coordinate system for each element.

1.3 Global Stiffness Matrix

In this truss we have three elements connected by 4 nodes whose connectivities are as below:

Element No.	Nodal Connectivity	
1	1	2
2	1	3
3	1	4

Firstly computing the stiffness matrix for **Element 1**, the local coordinate system of this element is at an angle of $90 + \alpha$ with respect to global coordinate system;

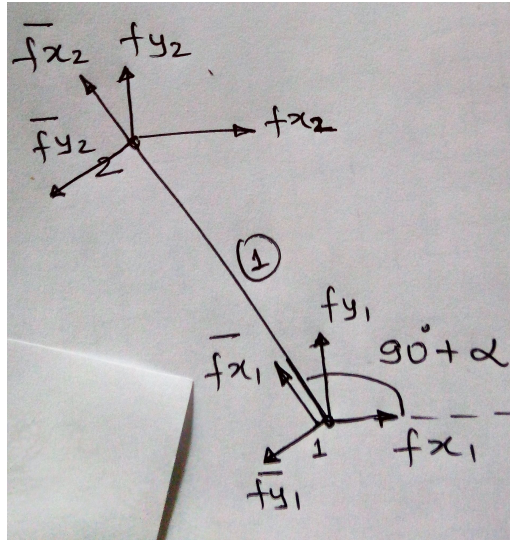


Figure 1.4: Element 1

$$K^1 = (T^1)^T \bar{K}^1 T^1 \quad (1.1)$$

where, K^1 is computed by taking into account the angle ϕ for that element here it is $\phi = 90 + \alpha$ hence $\cos(90 + \alpha) = -\sin\alpha$ and $\sin(90 + \alpha) = \cos\alpha$. It is given that $\cos \alpha = c$ and $\sin \alpha = s$. So we need to modify the entries of the elemental stiffness matrices from $\cos\phi$ and $\sin\phi$ to in terms of $\cos \alpha$ and $\sin\alpha$. We get K^1 as below :

$A^1 = A$ $E^1 = E$ and $L^1 = L/c$

$$K^1 = \frac{EAc}{L} \begin{bmatrix} s^2 & sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}$$

$$K^1 = \frac{EA}{L} \begin{bmatrix} s^2c & -sc^2 & -s^2c & sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -s^2c & sc^2 & s^2c & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{bmatrix}$$

Now computing the global stiffness matrix for **Element 2**, the local coordinate system of this element is at an angle of 90 with respect to global coordinate system;

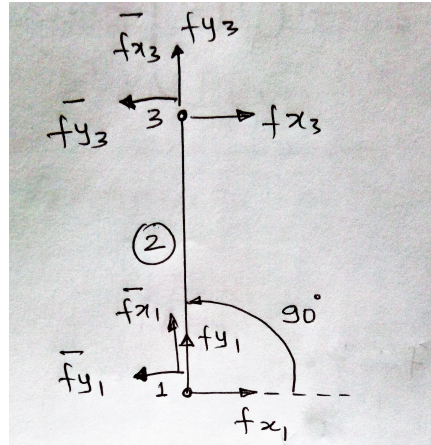


Figure 1.5: Element 2

$$K^2 = (T^2)^T \bar{K}^2 T^2 \quad (1.2)$$

where, K^2 is computed by taking into account the angle ϕ for that element here it is $\phi = 90$ hence $\cos(90) = 0$ and $\sin(90) = 1$. It is given that $\cos \alpha = c$ and $\sin \alpha = s$. So we need to modify the entries of the elemental stiffness matrices from $\cos\phi$ and $\sin\phi$ to in terms of $\cos \alpha$ and $\sin\alpha$. We get K^2 as below:

$A^2 = A$ $E^2 = E$ and $L^2 = L$

$$K^2 = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ -sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & -sc & s^2 \end{bmatrix}$$

$$K^2 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Now computing the global stiffness matrix for **Element 3**, the local coordinate system of this element is at an angle of 90 with respect to global coordinate system;

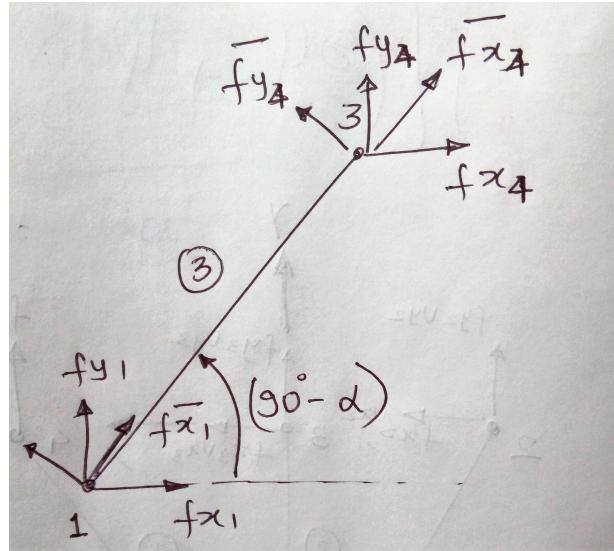


Figure 1.6: Element 1

$$K^3 = (T^3)^T \bar{K}^3 T^3 \quad (1.3)$$

where, K^3 is computed by taking into account the angle ϕ for that element here it is $\phi = 90 - \alpha$ hence $\cos(90 - \alpha) = \sin\alpha$ and $\sin(90 - \alpha) = \cos\alpha$. It is given that $\cos \alpha = c$ and $\sin \alpha = s$. So we need to modify the entries of the elemental stiffness matrices from $\cos\phi$ and $\sin\phi$ to in terms of $\cos \alpha$ and $\sin\alpha$. We get K^3 as below:

$$A^3 = A \quad E^3 = E \quad \text{and} \quad L^3 = L/c$$

$$K^3 = \frac{EAc}{L} \begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix}$$

$$K^3 = \frac{EA}{L} \begin{bmatrix} s^2c & sc^2 & -s^2c & -sc^2 \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -s^2c & -sc^2 & s^2c & sc^2 \\ -sc^2 & -c^3 & sc^2 & c^3 \end{bmatrix}$$

Now we will compute the global stiffness matrix from all the elemental stiffness matrix. This is done by adding the individual matrix entries of each element in their respective positions in the global matrix. This matrix would be a 8 X 8 matrix since the DOF of this system is 8. This matrix when computed is as below:

$$K = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & cs^2 & 0 & 0 & -c^2s & -c^2s \\ & 1 + 2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & \text{symm.} & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \quad (1.4)$$

1.4 Global Displacement and force vectors

The global displacement vector in most cases would be our vector of unknowns for a truss problem. Certain constraints on some nodes so as to we can solve the obtained system of equations. The Dirichlet boundary conditions i.e. the constraints on some node or nodes of the system on application would give us a unique solution to the problem. The Displacement vector for this given truss is as below:

$$U^T = [u_{x1} \quad u_{y1} \quad u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3} \quad u_{x4} \quad u_{y4}]$$

In this case the nodes 2, 3 and 4 are fixed i.e. they have zero displacement. We can say that $u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$ The global force vector has to be calculated element by element by using the relation below:

$$f^e = (T^e)^T \bar{f}^e \quad (1.6)$$

In our case the we have forces on node 1, a force of H units in the +ve X direction and force of P units in the -ve Y direction. Since the forces in this truss are coaxial with our global coordinate system we can write the global force vector as below:

$$f^T = [f_{x1} \quad f_{y1} \quad f_{x2} \quad f_{y2} \quad f_{x3} \quad f_{y3} \quad f_{x4} \quad f_{y4}]$$

1.5 Solving the system

In general our global system for this truss will be as below:

$$f = KU \quad (1.8)$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & cs^2 & 0 & 0 & -c^2s & -c^2s \\ & 1 + 2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \quad (1.9)$$

Applying the Displacement and Force Boundary conditions i.e.;

$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$ and $f_{x1} = H$ $f_{y1} = -P$ By applying the above conditions the equation will reduce to a 2 X 2 matrix as we can strike out rows and

columns corresponding to node 2, 3 and 4 as they are all fixed. We have zero displacements at these nodes. The reduced system of equations will be as below:

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1 + 2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} \quad (1.10)$$

Solving this system of equations we get $u_{x1} = \frac{HL}{2cs^2EA}$ and $u_{y1} = \frac{-PL}{EA(1+2c^3)}$

Case 1 : $\alpha \rightarrow 0$

$u_{x1} \rightarrow \infty$ and $u_{y1} = \frac{-PL}{3EA}$

This case would be equivalent to having all 3 truss elements fixed at one node and the two loads being applied at node 1. This would mean that the 3 bars are coincident. This can be witnessed from the fact that $u_{y1} = \frac{-PL}{3EA}$, i.e. all the three bars are contributing to the stiffness of element 1-3. The reason why u_{x1} "blows up" because doing this value of α would mean that it is a single bar rotating freely around node 3.

Case 2 : $\alpha \rightarrow \frac{\pi}{2}$

$u_{x1} \rightarrow \infty$ and $u_{y1} = \frac{-PL}{EA}$

This case would be equivalent to having all 3 truss elements fixed at one node and the two loads being applied at node 1. This would mean that the 2 bars (element 1 and 3) is completely fixed. The fact that $u_{y1} = \frac{-PL}{EA}$, i.e. only one bar is contributing to the stiffness of element 1-3. The presence of the two other bars has meaning in this case. The reason why u_{x1} "blows up" because doing this value of α would mean that it is a single bar rotating freely around node 3.

1.6 Computation of Axial Forces

Now we know all the unknown displacements for this structure. So by using these values we can obtain the axial forces in all the truss members.

Element 1

$$\bar{u}^1 = T^1 u^1 \quad (1.11)$$

$$\bar{u}^1 = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} \quad (1.12)$$

$$\bar{u}^1 = \frac{L}{EA} \begin{bmatrix} \frac{-H}{2cs} + \frac{-Pc}{(1+2c^3)} \\ \frac{-H}{2s^2} + \frac{Ps}{(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} \quad (1.13)$$

From the above displacement vector we get:

$$d_1 = \bar{u}^{x2} - \bar{u}^{x1} = \frac{L}{EA} \left(\frac{H}{2cs} + \frac{Pc}{(1+2c^3)} \right) \quad (1.14)$$

$$F^1 = \frac{EA c}{L} d_1 \quad (1.15)$$

$$F^1 = \left(\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)} \right) \quad (1.16)$$

Element 2

$$\bar{u}^2 = T^2 u^2 \quad (1.17)$$

$$\bar{u}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} \quad (1.18)$$

$$\bar{u}^2 = \frac{L}{EA} \begin{bmatrix} \frac{-P}{(1+2c^3)} \\ \frac{-H}{2cs^2} \\ 0 \\ 0 \end{bmatrix} \quad (1.19)$$

From the above displacement vector we get:

$$d_2 = \bar{u}^{x4} - \bar{u}^{x1} = \frac{L}{EA} \left(\frac{P}{(1+2c^3)} \right) \quad (1.20)$$

$$F^2 = \frac{EA}{L} d_2 \quad (1.21)$$

$$F^2 = \frac{P}{(1+2c^3)} \quad (1.22)$$

Element 3

$$\bar{u}^3 = T^3 u^3 \quad (1.23)$$

$$\bar{u}^3 = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} \frac{HL}{2cs^2EA} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} \quad (1.24)$$

$$\bar{u}^3 = \frac{L}{EA} \begin{bmatrix} \frac{H}{2cs} + \frac{-Pc}{(1+2c^3)} \\ \frac{-H}{2s^2} + \frac{-Ps}{(1+2c^3)} \\ 0 \\ 0 \end{bmatrix} \quad (1.25)$$

From the above displacement vector we get:

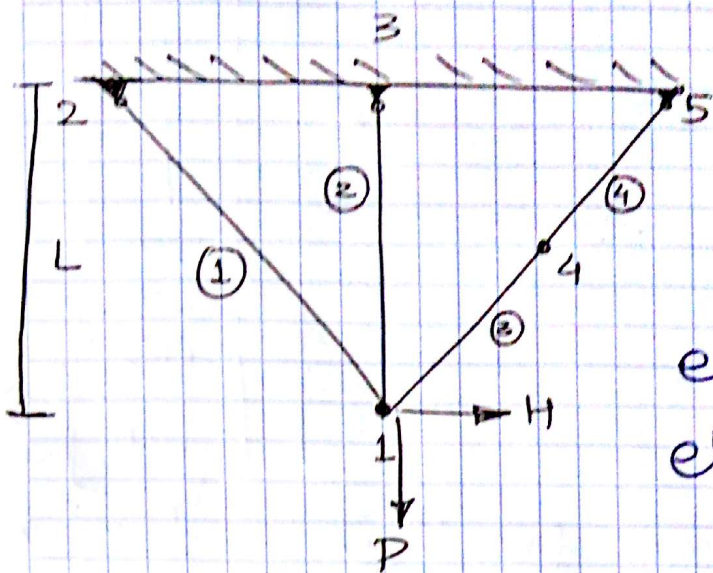
$$d_3 = \bar{u}^{x3} - \bar{u}^{x1} = \frac{L}{EA} \left(\frac{-H}{2cs} + \frac{Pc}{(1+2c^3)} \right) \quad (1.26)$$

$$F^3 = \frac{EA c}{L} d_3 \quad (1.27)$$

$$F^3 = \frac{-H}{2s} + \frac{Pc^2}{(1+2c^3)} \quad (1.28)$$

The reactions F^1 and F^3 would blow up as $\alpha \rightarrow 0$ because the three bars would have all the three nodes 2 3 and 4 coincident. This would mean that all the reaction is now from a single node. so instead of having 3 fixed nodes we will have a single fixed node taking all the load i.e. Node 3.

Assignment 2



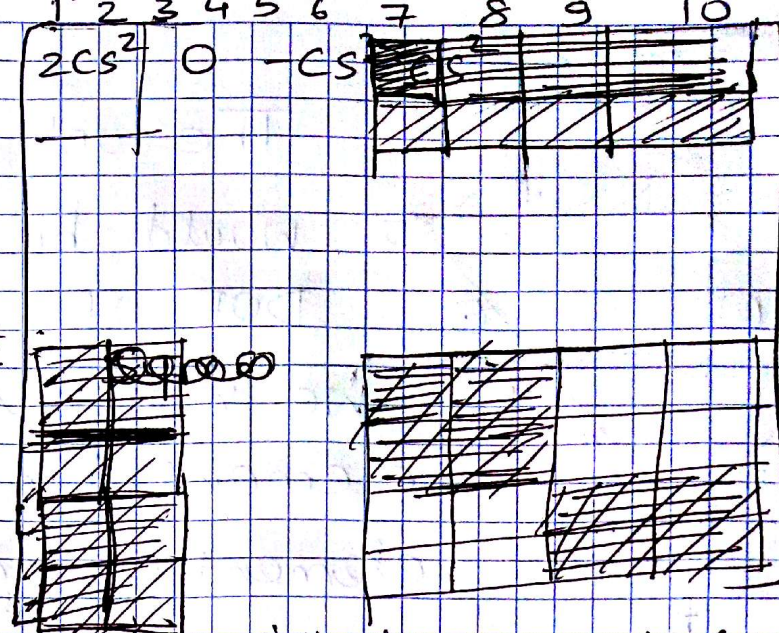
The only change would be added DOF of system. We will have new elemental matrices for element (3) and (4).

k^1, k^2 will be the same as previous case. k^3 for the previous case will be replaced by new k^3 and k^4 due to presence of a node in between the element.

$$k^3 = \frac{EA(2)}{L} \begin{bmatrix} 1 & 2 & 7 & 8 \\ s^2c & sc & -s^2 & -sc \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -s^2c & -sc^2 & s^2c & sc^2 \\ -sc^2 & -c^3 & cs^2 & c^3 \end{bmatrix}$$

$$k^4 = \frac{EA(2)}{L} \begin{bmatrix} 1 & 2 & 9 & 10 \\ | & | & | & | \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 9 \\ 10 \end{matrix}$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ \vdots \\ f_{x4} \\ f_{y4} \\ f_{x5} \\ f_{y5} \end{bmatrix} = \frac{EA}{L}$$



$$\begin{bmatrix} u_{x1} \\ u_{y1} \\ \vdots \\ u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \end{bmatrix}$$

No of unknowns will inc for this fixed system u_{x4} and $u_{y4} \rightarrow$ added unknowns.

~~conclusion~~

The master stiffness would be singular as it wont have a unique solution. The entries will go in the above shaded regions of the matrix. Physically, this singularity would imply that the truss has now become a "MECHANISM". If we increase nodes in this truss the DOF will increase and it would form a more complex mechanism.

Since, Node 5 is fixed $u_{x5} = u_{y5} = 0$ we can ignore that row column from the stiffness matrix.

CONCLUSIONS

- ∴ More the number of nodes \Rightarrow more DOF
- more DOF \Rightarrow Mechanism
- ∴ Matrix singular \Rightarrow No unique solution.