Prove: $\begin{cases} \frac{\partial \rho \bar{v}}{\partial t} + \nabla\cdot(\rho \bar{v}\otimes\bar{v}) - \nabla\cdot\underset{\sim}{\sigma} = \rho\bar{b} & (1a) \\ \nabla\cdot\bar{v} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - \nu\nabla\cdot(\nabla\bar{v}) + \nabla p^* = \bar{b} & (2a) \\ \nabla\cdot\bar{v} = 0 \end{cases}$

$\rho$ is constant

$$(1a) \Rightarrow \frac{\partial \bar{v}}{\partial t} + \nabla\cdot(\bar{v}\otimes\bar{v}) - \nabla\cdot\frac{1}{\rho}\underset{\sim}{\sigma} = \bar{b} \qquad (3)$$

Newtonian Fluid: $\underset{\sim}{\sigma} = -p\underset{\sim}{I} + \mu\left[\nabla\bar{v} + (\nabla\bar{v})^T\right]$

$$\nabla\cdot\underset{\sim}{\sigma} = -\nabla p + \mu\left[\nabla\cdot\nabla\bar{v} + \nabla\cdot(\nabla\bar{v})^T\right]$$

$$= -\nabla p + \mu\,\nabla\cdot\nabla\bar{v} \quad \text{with } \nabla\cdot(\nabla\bar{v})^T = \nabla(\nabla\cdot\bar{v}) = \bar{0} \qquad (4)$$

Substitute (4) into (3) we have.

$$\frac{\partial \bar{v}}{\partial t} + \nabla\cdot(\bar{v}\otimes\bar{v}) - \nu\,\nabla\cdot(\nabla\bar{v}) + \nabla p^* = \bar{b} \quad \text{where } p^* = \frac{p}{\rho}$$

$$\nabla\cdot(\bar{v}\otimes\bar{v}) = (v_i v_j)_{,j} = v_{i,j}v_j + v_i \cancel{v_{j,j}}^{\,0} = (\bar{v}\cdot\nabla)\bar{v}$$

Hence it is equivalent to

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - \nu\nabla\cdot(\nabla\bar{v}) + \nabla p^* = \bar{b} \qquad \square$$