Finite Elements in Fluids, Assignment 6, Unsteady Navier Stokes

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Part 1. On Monolithic Schemes:

-Implicit Second Order Monolithic Scheme : Crank-Nicolson
Navier Stoked equations can be written in the base form of Θ family as:

$$\frac{v^{n+1}-v^n}{\Delta t} - \Theta(f - v\cdot\nabla)v + \nu\nabla^2 v)^{n+1} - \nabla p^{n+1} = [1-\Theta](f - (v\cdot\nabla)v + \nu\nabla^2 v)^n$$

$$\nabla\cdot v^{n+1} = 0$$

By selecting $\Theta = 1/2$, and rearranging:

$$\frac{1}{\Delta t}v^{n+1} + \frac{1}{2}(v^{n+1}\cdot\nabla)v^{n+1} - \frac{1}{2}\nu\nabla^2 v^{n+1} + \frac{1}{2}\nabla p^{n+1} = \frac{1}{2}f^n + f^{n+1} - \frac{1}{2}(v^n\cdot\nabla)v^n + \frac{1}{2}\nu\nabla^2 v^n + \frac{1}{\Delta t}v^n$$

$$\nabla\cdot v^{n+1} = 0$$

Which leads to the following system of equations:

$$\begin{bmatrix} \frac{1}{\Delta t}\boldsymbol{M} + \frac{1}{2}(\boldsymbol{C}(\boldsymbol{v}^{n+1}) + \boldsymbol{K}) & \boldsymbol{G}^T \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{n+1} - \frac{1}{2}\boldsymbol{C}(\boldsymbol{v}^n)\boldsymbol{v}^n - \frac{1}{2}\boldsymbol{K}\boldsymbol{v}^n + \frac{1}{\Delta t}\boldsymbol{M}\boldsymbol{v}^n \\ \boldsymbol{0} \end{bmatrix}$$

Notice that for the specific case of the cavity problem, the source term $\boldsymbol{f}$ is evaluated at the current time step only.

Crank-Nicolson Scheme is second order implicit, which is unconditionally stable, but special care has to be taken regarding the convective term. In order to deal with it, Newton-Raphson method is employed.

Newton-Raphson

The first step for applying Newton-Raphson is to obtain the residual:

$$\boldsymbol{r} = \begin{bmatrix} \left(\frac{1}{\Delta t}\boldsymbol{M} + \frac{1}{2}(\boldsymbol{K} + \boldsymbol{C}(\boldsymbol{v}^{n+1}))\right)\boldsymbol{v}^{n+1} + \boldsymbol{G}^T\boldsymbol{p}^{n+1} - \boldsymbol{f}^{n+1} + \frac{1}{2}\boldsymbol{C}(\boldsymbol{v}^n)\boldsymbol{v}^n + \frac{1}{2}\boldsymbol{K}\boldsymbol{v}^n - \frac{1}{\Delta t}\boldsymbol{M}\boldsymbol{v}^n \\ \boldsymbol{G}\boldsymbol{v}^{n+1} \end{bmatrix}$$

By taking the derivative of the components of $\boldsymbol{r}$, one obtains the Jacobian Matrix $\boldsymbol{J}$ as:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial r_1}{\partial \boldsymbol{v}^{n+1}} & \frac{\partial r_1}{\partial \boldsymbol{p}^{n+1}} \\ \frac{dr_2}{d\boldsymbol{v}^{n+1}} & \frac{dr_1}{d\boldsymbol{p}^{n+1}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\boldsymbol{M} + \frac{1}{2}(\boldsymbol{C}(\boldsymbol{v}^{n+1}) + \boldsymbol{K}) + \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{v}^{n+1}} & \boldsymbol{G}^T \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix}$$

Recalling that:

$$\boldsymbol{C} = \int_\Omega \boldsymbol{w}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}d\Omega$$

The simplest way to obtain the discretization is to select the interior v to be evaluated at the Gauss point, and the exterior one to be obtained at the nodes:

$$\boldsymbol{C} = \left[matN\right]^T \begin{bmatrix} V_x & 0 & V_y & 0 \\ 0 & V_x & 0 & V_y \end{bmatrix} \left[gradN\right]$$

To obtain the derivative $d\boldsymbol{C}/d\boldsymbol{v}$, one could use the concept of the directional derivative:

$$D(\boldsymbol{w} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{v})[\delta v] = \boldsymbol{w} \cdot (\delta v \cdot \nabla)\boldsymbol{v} + \boldsymbol{w} \cdot (\boldsymbol{v} \cdot \nabla)\delta v$$

From this last expression, the second term leads to the Convection Matrix already described as:

$$\boldsymbol{w} \cdot (\boldsymbol{v} \cdot \nabla)\delta v = \begin{bmatrix} V_x \frac{\partial \delta V_x}{\partial x} & V_y \frac{\partial \delta V_x}{\partial y} \\ V_x \frac{\partial \delta V_y}{\partial x} & V_y \frac{\partial \delta V_x}{\partial y} \end{bmatrix} = \left[matN\right] \begin{bmatrix} V_x & 0 & V_y & 0 \\ 0 & V_x & 0 & V_y \end{bmatrix} \begin{bmatrix} \frac{\partial \delta V_x}{\partial x} \\ \frac{\partial \delta V_y}{\partial x} \\ \frac{\partial \delta V_x}{\partial y} \\ \frac{\partial \delta V_y}{\partial y} \end{bmatrix}$$

The first term, however, leads to the derivative $\partial \boldsymbol{C}/\partial \boldsymbol{v}$

$$(\delta v \cdot \nabla)\boldsymbol{v} = \left[matN\right]^T \begin{bmatrix} \delta V_x \frac{\partial V_x}{\partial x} & \delta V_y \frac{\partial V_x}{\partial y} \\ \delta V_x \frac{\partial V_y}{\partial x} & \delta V_y \frac{\partial V_x}{\partial y} \end{bmatrix}$$

So, the derivative, needed for Newton-Raphson method is:

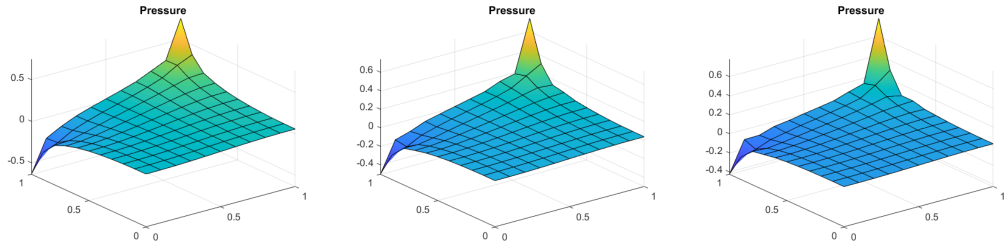
$$\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{v}^{\boldsymbol{n+1}}} = \left[matN\right]^T \begin{bmatrix} \frac{\partial \boldsymbol{v}_x}{dx} & \frac{\partial \boldsymbol{v}_x}{dy} \\ \frac{\partial \boldsymbol{v}_y}{dx} & \frac{\partial \boldsymbol{v}_y}{dy} \end{bmatrix} \left[matN\right]$$

The algorithm to solve the Navier Stokes Equations Using the Crank Nicolson Monolithic scheme is then:

Enter: Geometry, viscosity, and Boundary and Initial Conditions;
Obtain Matrices M, K, G, and vector f;
Build a Preliminary system A;
$\begin{bmatrix} (1/2)K + (1/dt\boldsymbol{M}) & \boldsymbol{G}^T \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix}$;
while *current time* < *final time* **do**
  Obtain $\boldsymbol{C}$ on last time step ;
  while *residual* > *Tolerance* **do**
    Obtain $\boldsymbol{C}$, $d\boldsymbol{C}/d\boldsymbol{v}$ on current time step;
    Complete matrix $\boldsymbol{A}$ by adding Convection Contribution;
    Obtain $\boldsymbol{J}$, $\boldsymbol{r}$ and $\boldsymbol{RHS}$;
    Calculate Current Step solution $\boldsymbol{v}^{n+1}$, $\boldsymbol{p}^{n+1}$;
    Check Convergence;
  end
  Go to next time step Update: $\boldsymbol{v}^n = \boldsymbol{v}^{n+1}$, $\boldsymbol{p}^n = \boldsymbol{v}^{p+1}$
end
Plot

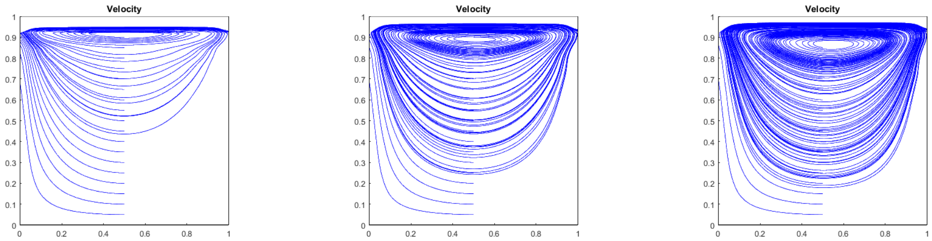
Algorithm 1: Basic Structure of the Crank Nicolson Scheme

The solutions obtained when running the Cavity Flow MATLAB code for a value of Reynolds number:100 and using LBB stable elements Q2Q1; are shown next:



(a) t=0.02 (b) t=0.1 (c) t=0.25

Figure 1: Crank Nicholson Results for Pressure Evolution



(a) t=0.02 (b) t=0.1 (c) t=0.25

Figure 2: Crank Nicholson Results for Velocity Evolution

Part 2. On Multistage Schemes:

Chorin Temam scheme splits the solution of the Navier Stokes equations into two steps. The first step deals with the velocity only, and the solution to it is an intermidiate velocity $\hat{\boldsymbol{v}}$.

First Step

$$\begin{cases} \frac{\hat{\boldsymbol{v}}^{n+1}-\boldsymbol{v}^{n}}{\Delta t} + (\hat{\boldsymbol{v}}^{n+1}\cdot\nabla)\hat{\boldsymbol{v}}^{n+1} - \nu\nabla^2\hat{\boldsymbol{v}}^{n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega \\ \nabla\hat{\boldsymbol{v}}^{n+1} = \boldsymbol{v}_D^{n+1} & \text{in } \partial\Omega \end{cases}$$

Discretizing the first step leads to the following nonlinear equation:

$$\left(\frac{1}{\Delta t}\boldsymbol{M} + \boldsymbol{C}(\hat{\boldsymbol{v}}^{n+1}) + \boldsymbol{K}\right)\hat{\boldsymbol{v}}^{n+1} = \boldsymbol{f}^{n+1} + \frac{1}{\Delta t}\boldsymbol{M}\boldsymbol{v}^{n}$$

The nonlinearity is resolved using Newton Raphson again, for which the residual $\boldsymbol{r}$ and the Jacobian matrix $\boldsymbol{J}$ are given as:

$$\boldsymbol{r} = \left(\frac{1}{\Delta t}\boldsymbol{M} + (\boldsymbol{K} + \boldsymbol{C}(\hat{\boldsymbol{v}}^{n+1}))\right)\boldsymbol{v}^{n+1} - \boldsymbol{f}^{n+1} - \frac{1}{\Delta t}\boldsymbol{M}\boldsymbol{v}^{n}$$

$$\boldsymbol{J} = \frac{1}{\Delta t}\boldsymbol{M} + (\boldsymbol{K} + \boldsymbol{C}(\hat{\boldsymbol{v}}^{n+1})) + \frac{\partial\boldsymbol{C}}{\partial\hat{\boldsymbol{v}}^{\boldsymbol{n+1}}}$$

Second Step

The second step of Chorin Temam scheme does incorporate the pressure term. It also obtains the end of step velocity $\boldsymbol{v}^{n+1}$, which is the variable of interest, unlike $\hat{\boldsymbol{v}}^{n+1}$.

$$\begin{cases} \frac{\boldsymbol{v}^{n+1}-\hat{\boldsymbol{v}}^{n+1}}{\Delta t} + \nabla\boldsymbol{p}^{n+1} = \boldsymbol{0} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{v}^{n+1} = \boldsymbol{v}_D^{n+1} & \text{in } \partial\Omega \end{cases}$$

Discretizing the second step leads to the following system of quations:

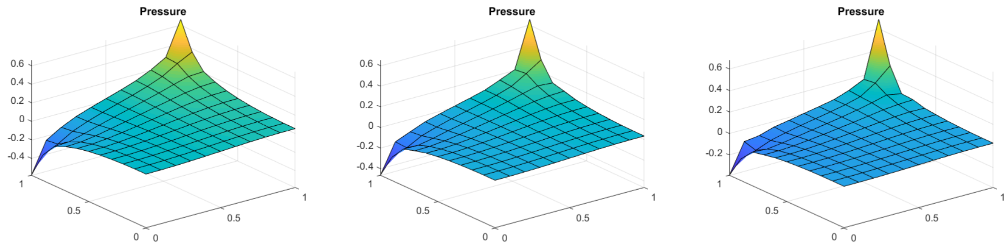
$$\begin{bmatrix} \frac{1}{\Delta t}\boldsymbol{M} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{v}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\boldsymbol{M}\hat{\boldsymbol{v}}^{n+1} \\ \boldsymbol{0} \end{bmatrix}$$

The algorithm for solving the transient Navier Stokes equations using Chorin Temam scheme is then:

Enter: Geometry, viscosity, and Boundary and Initial Conditions;
Obtain Matrices M, K, G, and vector f;
Build a Preliminary term A;
$\left[\boldsymbol{K} + (1/dt\boldsymbol{M})\right]$;
while *current time* < *final time* **do**
    Obtain $\boldsymbol{C}$ on last time step ;
    FIRST STEP;
    while *residual* > *Tolerance* **do**
        Obtain $\boldsymbol{C}$, $d\boldsymbol{C}/d\boldsymbol{v}$ on current step intermidiate velocity;
        Complete term $\boldsymbol{A}$ by adding Convection Contribution;
        Obtain $\boldsymbol{J}$, $\boldsymbol{r}$ and $\boldsymbol{RHS}$;
        Calculate Current Step intermediate velocity $\hat{\boldsymbol{v}}^{n+1}$;
        Check Convergence;
    end
    SECOND STEP;
    solve the system $\begin{bmatrix} 1/dt\boldsymbol{M} & \boldsymbol{G}^T \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix} = \begin{bmatrix} 1/dt\boldsymbol{M}\hat{\boldsymbol{v}}^{n+1} \\ \boldsymbol{0} \end{bmatrix}$;
    Update: $\boldsymbol{v}^n = \boldsymbol{v}^{n+1}$, $\boldsymbol{p}^n = \boldsymbol{v}^{p+1}$;
    Go to next time step;
end
Plot

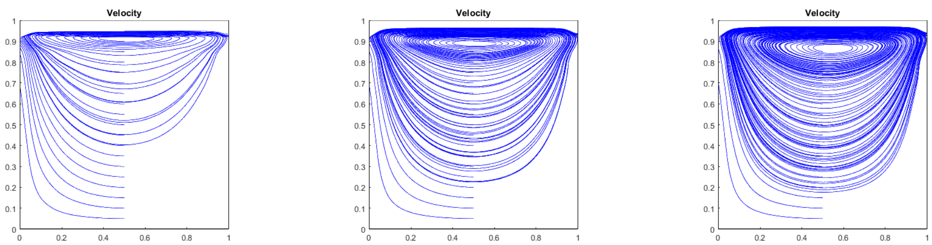
Algorithm 2: Basic Structure of the Chorin Themam Scheme

The solutions obtained when running the Cavity Flow MATLAB code for a value of Reynolds number:100 and using LBB stable elements Q2Q1; are shown next:



(a) t=0.02 (b) t=0.1 (c) t=0.25

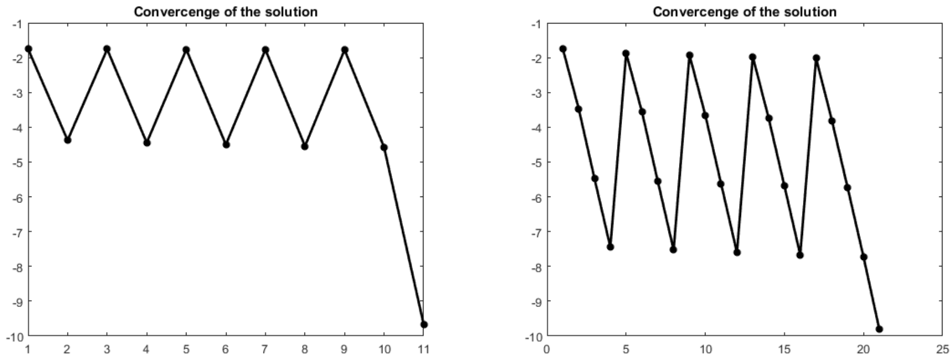
Figure 3: Chorin Temam Results for Pressure Evolution



(a) t=0.02 (b) t=0.1 (c) t=0.25

Figure 4: Chorin Temam Results for Velocity Evolution

One can observe the number of iterations needed for a solution to be obtained using either scheme.



(a) Chorin Temam

(b) Crank Nicolson

Figure 5: Convergence comparison for both schemes, Chorin Temam and Crank Nicolson

Conclusions:

Both schemes are solving the same equations and both are using the Newton Raphson method to resolve the nonlinearity. However, it is interesting to notice that the results are achieved faster using Chorin Temam. If one looks at the number of iterations to reach convergence, it is possible to see that convergence is achieved on 2 iterations for Chorin Temam, while it takes 4 for Crank Nicolson. This is because the pressure is usually the variable that drives convergence speed. By treating both variables separately, through an intermediate velocity, Chorin Temam is more efficient.