FEF Exam
Mid-Term

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Computational Mechanics

Ex-1

$$u_t + u_{xx} = 0 \quad -① $$

$u = u_0(x)$

$u = 1$

$$u_0(x) = \begin{cases} 1 & , x \in [0, p) \\ 1 - \frac{x-p}{q-p} & , x \in [p, q] \\ 0 & , x \in (q, 1] \end{cases}$$

$p = 0.64$ and $q = 0.84$

$$f_x(u) = u\,u_x \quad -②$$

$$u_t + f_x(u) = 0 \quad \text{(Cauchy eq)} \quad -③$$
$$u = u_0(x)$$

Taylor series
$$u^{n+1} = u^n + \Delta t\, u_t^n + \frac{(\Delta t)^2}{2} u_{tt}^n + \underbrace{O(\Delta t)^3}_{\text{Truncation error}}$$

$$\frac{u^{n+1} - u^n}{\Delta t} = u_t^n + \frac{\Delta t}{2} u_{tt}^n + \underbrace{O(\Delta t)^2}_{\text{Neglect this error}}$$

We get
$$\frac{u^{n+1} - u^n}{\Delta t} = u_t^n + \frac{\Delta t}{2} u_{tt}^n \quad -④$$

$$u_t = -f_x \quad \text{(From cauchy)} \quad -⑤$$

$$u_{tt} = -f_{xt} = -f_{tx} = -(f_u u_t)_x$$
$$= (u^2 u_x)_x$$

Burger Eq. Time discretisation
$$\frac{u^{n+1} - u^n}{\Delta t} = -f_x^n + \frac{\Delta t}{2} \left((u^n)^2 u_x^n\right)_x$$

Now, integration by parts of spatial terms TG-2 formulation is.

$$\int_0^L w\left(\frac{u^{n+1} - u^n}{\Delta t}\right) dx = \int_0^L w_x f^n dx - \frac{\Delta t}{2}\int_0^L w_x (u^n)^2 u_x^n \, dx - \left[w f^n - w\frac{\Delta t}{2}(u^n)^2 u_x^n\right]_0^{x=L}$$

$$f_t = \frac{df}{du}\frac{\partial u}{\partial t} = -u^2 u_x$$

$\therefore$ ~~g^{th}~~ term of R.H.S, i.e.,

$$-\left[-w\frac{\Delta t}{2}(u^n)^2 u_x^n\right]_{x=0}^{x=L} = -\left[w\frac{\Delta t}{2} f_t^n\right]_0^L$$

<u>Now</u> Inlet Boundary Cond^n is

$$f(0,t) = \left(\frac{u^2}{2}\right)_{x=0} = \frac{1}{2} \quad [u(0,t) = 1]$$

3. Classical representation (2-pt gaussian quadrature)

~~$\sum N_i$~~

$$f = \sum f(N_i u_i)$$

Group Representation (Interpolation at element nodes)

$$f = \sum N_i f(u_i)$$

Ex-2 $\quad -\nu u_{xx} + \beta u_x = 0 \quad , x \in (-1,1) \quad$ — ①

$u = 0 \quad , \; x \in -1$

$u = -1 \quad , \; x \in 1$

$\nu = 0.03 \qquad \beta = 1.8$ (convection vel)

Sol^n Here $s = 0$ (source term)

Integrating eq ① and multiply by weights, w

$$\underbrace{\int_\Omega w \beta u_x}_{\text{I}} + \underbrace{\int_\Omega w \nu u_{xx}}_{\text{II}} = 0 \quad — ②$$

$\cancel{\int w \nu u_{xx} = \int w \nu u_x}$

II term can be written as

$-\int w \nabla \cdot (\nu \nabla u) =$

using divergence theorem

$$\nabla \cdot (w \cdot (\nu \nabla u)) = \nabla w \cdot (\nu \nabla u) + \underline{w \nabla \cdot (\nu \nabla u)}$$

$$\Rightarrow -\int w \nabla \cdot (\nu \nabla u) = \int \nabla w \cdot (\nu \nabla u) - \int \nabla \cdot (w \cdot (\nu \nabla u))$$

$w = 0$ for Γ_0

$$\Rightarrow -\int w \nabla \cdot (\nu \nabla u) = \int \nabla w \cdot \nu \nabla u$$

$$= \int w_x \nu u_x \quad — ③$$

∴ Eq ② becomes

$$\int_{x=-1}^{x=1} w \beta u_x dx + \int_{x=-1}^{x=1} w_x \nu u_x \, dx = 0$$

$$a(w, u) + c(w, \beta, u) = 0$$

$\cancel{\int \beta N_A \frac{\partial N_B}{\partial x} + \nu}$

$$\int_0^L \sum_{B=2}^{n_{eq}+1} \left(\beta N_A \frac{\partial N_B}{\partial x} + \nu \frac{\partial N_A}{\partial x} \frac{\partial N_B}{\partial x} \right) u_B \, dx = 0$$

$$\text{Nodes} = \{ 1, n_{nod} \}$$

$$= \{ 1, 2, 3, - \text{—}, n_{eq}, n_{eq+1}, n_{np} \}$$

$$(C + K)\, u = f$$

$$N_1 = \frac{1}{2}(1-\xi) ; \quad N_2 = \frac{1}{2}(1+\xi) ; \quad dx = \frac{(x_2 - x_1)}{2} d\xi = \frac{h}{2} d\xi$$

- Element convection matrix

$$C^e = \beta \int_{\Omega^{(e)}} \begin{pmatrix} N_1 \frac{\partial N_1}{\partial x} & N_1 \frac{\partial N_2}{\partial x} \\ N_2 \frac{\partial N_1}{\partial x} & N_2 \frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{\beta}{2} \begin{pmatrix} -1 & +1 \\ 1 & +1 \end{pmatrix}$$

- Element diffusion matrix

$$K^{(e)} = \nu \int_{\Omega^{(e)}} \begin{pmatrix} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} \end{pmatrix} dx$$

$$= \frac{\nu}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

b) There will be oscillations, in the eq conv-diff eq, to reduce, we introduce stabilisation term

Stabilisation term

$$\sum_e \int P(w)\, \tau\, R(u)$$

$$P(w) = \beta\cdot\nabla w + \nabla\cdot(\gamma\nabla w)$$
$$= \beta w_x + \gamma w_{xx}$$

$$R(u) = \beta\cdot\nabla u - \nabla\cdot(\gamma\nabla u) - \cancel{s}^{\,0}$$
$$= \beta u_x - \gamma u_{xx}$$

∴ Stabilisation term is

$$\sum_e \int (\beta w_x + \gamma w_{xx})\, \tau\, \underbrace{(\beta u_x - \gamma u_{xx})}_{L(u)}$$

Discritised form

$$a(w,u) + c(w,\beta,u) + \sum_e \int_{\Omega^e} (\beta w_x + \gamma w_{xx})\tau (\beta u_x - \gamma u_x) = 0$$

$$\int_{-1}^{+1} \gamma \frac{\partial N_A}{\partial x}\frac{\partial N_B}{\partial x} u_B + \int_{-1}^{+1} \beta N_A \frac{\partial N_B}{\partial x} u_B +$$

$$\sum_e \int_{\Omega^{(e)}} \left(\beta \frac{\partial N_A}{\partial x} + \gamma \frac{\partial^2 N_A}{\partial x^2}\right) \tau \left(\beta \frac{\partial N_B}{\partial x} - \gamma \frac{\partial N_B}{\partial x}\right) u_B = 0$$

(c) SGS have no instabilities like SUPG and GLS

(C) Stabilization term:
for linear element diffusion term is zero.
For quadratic element $\gamma \neq 0$

For 1-D linear elements superconvergence is obtained.

τ vanishes when mesh is refined.
Convergence depends on asymptotic behaviour of τ

However, 4th order accuracy is not reached for higher dimensions.

$$\tau = \frac{h}{2\beta}\left(\coth Pe - \frac{1}{Pe}\right)$$