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FEF Final Exam

1.) Stokes problem solved with Q_2Q_1 elements.

a) The Q_2Q_1 pair is LBB stable, and therefore stability is guaranteed. This means that there is a unique solution to the algebraic problem resulting from the FE discretization. In conclusion, it is a suitable pair to discretise the equation.

b) It is not necessary to use a stabilized formulation for this problem since satisfying the LBB condition is a sufficient condition to guarantee the stability of the solution.

c) In the case of a DG method, the discretised system will pass the LBB condition even if we use the same degree of interpolation for pressure and velocity. Therefore, we may use a polynomial approximation of degree 6 for pressure.

Assuming a 2D problem, we will have two velocity degrees of freedom and one pressure degree of freedom for every node on the inside of each domain. The nodes at the interface between domains (Γ) will duplicate, with two velocities and a pressure for each of them.

d) For a HDG discretization, the global problem requires only the solution of the hybrid variable $\hat{u}$ at interface nodes, which are not duplicated along the edges of the elements like in the previous method, but only at the vertices where more than two elements meet.

The local problem requires the solution of the hybrid variable Le, the velocities and the pressure on all nodes inside each element, not including the nodes at the edges, where $\hat{u}$ is imposed as a dirichlet BC.

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e.) 1. Break the domain into n_{el} elements

2. Introduce mixed variable to rewrite the equations as first order equations

3. Introduce hybrid variable as a Dirichlet BC in al local problems and solve each local problem in terms of hybrid variable as a discrete system of equations

4. Impose transmission conditions between elements

5. Solve global problem for hybrid variable

6. Postprocess → Calculate variables which were initially solved in terms of the previously unknown hybrid variable in the local problems.

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2.) Navier Stokes Problem with reaction term σu, $\sigma > 0$

$Q_1 Q_1$ Discretization

$$\begin{cases} \dfrac{\partial u}{\partial t} \underbrace{- \nu \nabla^2 u}_{K(u)} + \underbrace{(u \cdot \nabla) u}_{C(u)} + \sigma u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u = u_0 & \text{in } \Omega,\ t = 0 \end{cases}$$

a.) Crank-Nicolson time discretization $\longrightarrow \theta = \frac{1}{2}$

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + C(u^{n+\theta}) + K(u^{n+\theta}) + \sigma u^{n+\theta} + \nabla p^{n+1} = f^{n+\theta} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

(no time derivative associated with pressure)

$$\begin{cases} u^{n+1} + \Delta t\, \theta \left[C(u^{n+1}) + K(u^{n+1}) + \sigma u^{n+1} \right] + \Delta t\, \nabla p^{n+1} \\ \qquad = u^n + \Delta t \left[\theta f^{n+1} + (1-\theta) f^n + (\theta - 1)\left(C(u^n) + K(u^n) + \sigma u^n \right) \right] \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

Applying $\theta = \frac{1}{2}$:

$$\begin{cases} u^{n+1} + \frac{\Delta t}{2} \left[C(u^{n+1}) + K(u^{n+1}) + \sigma u^{n+1} \right] + \Delta t\, \nabla p^{n+1} \\ \qquad = u^n + \frac{\Delta t}{2} (f^{n+1} + f^n) - \frac{\Delta t}{2} \left(C(u^n) + K(u^n) + \sigma u^n \right) \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

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b) weak form

weighting function for velocity $\rightarrow w \in \mathcal{V}$

weighting function for pressure $\rightarrow q \in \mathcal{L}$

$$\mathcal{V} = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D \}$$

$$\mathcal{L} = \{ p \in L^2(\Omega) \}$$

LHS
$$\int_\Omega w\, u^{n+1}\, d\Omega + \frac{\Delta t}{2}\left[\int_\Omega w (u^{n+1}\cdot\nabla) u^{n+1}\, d\Omega - \int_\Omega w\cdot(\nu \nabla^2 u^{n+1})\, d\Omega + \int_\Omega w\cdot(\sigma u^{n+1})\, d\Omega\right] + \Delta t \int_\Omega w\cdot \nabla p^{n+1}\, d\Omega$$

RHS
$$= \int_\Omega w\cdot u^n\, d\Omega + \frac{\Delta t}{2}\left[\int_\Omega w (f^{n+1} + f^n)\, d\Omega - \int_\Omega w (u^n\cdot\nabla) u^n\, d\Omega + \int_\Omega w\cdot(\nu\nabla^2 u^n)\, d\Omega - \int_\Omega w\cdot(\sigma u^n)\, d\Omega\right]$$

$$\int_\Omega q\, \nabla\cdot u\, d\Omega = 0$$

Integrating by parts: (terms at Γ_D resulting from integration by parts are neglected since $w = 0$ at Γ_D)

$$\begin{cases} \int_\Omega w\, u^{n+1}\, d\Omega + \frac{\Delta t}{2}\left[\int_\Omega w (u^{n+1}\cdot\nabla) u^{n+1}\, d\Omega + \int_\Omega \nabla w : (\nu \nabla u^{n+1})\, d\Omega + \int_\Omega w (\sigma u^{n+1})\, d\Omega\right] - \Delta t \int_\Omega (\nabla\cdot w) p^{n+1}\, d\Omega \\ \quad = \int_\Omega w\cdot u^n\, d\Omega + \frac{\Delta t}{2}\left[\int_\Omega w (f^{n+1} + f^n)\, d\Omega - \int_\Omega w (u^n\cdot\nabla) u^n\, d\Omega - \int_\Omega \nabla w : (\nu\nabla u^n)\, d\Omega - \int_\Omega w\cdot(\sigma u^n)\, d\Omega\right] \\ \int_\Omega q\, \nabla\cdot u\, d\Omega = 0 \end{cases}$$

Rewriting in a more compact notation:

$$\boxed{\begin{cases} (w, u^{n+1}) + \frac{\Delta t}{2}\left[c(u^{n+1}; w, u^{n+1}) + a(w, u^{n+1}) + \sigma(w, u^{n+1})\right] - \Delta t\, b(w, p^{n+1}) \\ \quad = (w, u^n) + \frac{\Delta t}{2}\left[(w, f^{n+1}) + (w, f^n) - c(u^n; w, u^n) - a(w, u^n) - \sigma(w, u^n)\right] \\ b(u, q) = 0 \end{cases}}$$

Since $Q_1 Q_1$ is not LBB stable, we must also add stabilization terms. For the Navier-Stokes problem, we may use the stabilized formulation proposed by Tezduyar and Osawa.

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c) FE discretization

$u^h = v^h + u_D^h \longrightarrow v^h = \sum N_i(x)\, u_i \qquad u_D^h = \sum N_i(x)\, u_{Di}$

$p^h = \sum \hat{N}_i\, p_i \qquad\qquad u^h \in S^h$

$w^h = \text{span}\{N\} \qquad q^h = \text{span}\{\hat{N}\} \qquad\qquad p^h \in \mathcal{L}^h$

Since we are using a Q_1Q_1 approximation $\longrightarrow N_i = \hat{N}_i$

Now we define the following matrices:

$$M = \int_\Omega N_i N_j \, d\Omega \qquad K = \int_\Omega \nabla N_i \cdot (\nu \nabla N_j)\, d\Omega$$

$$C(u) = \int_\Omega N_i (u \cdot \nabla) N_j \, d\Omega \qquad G = -\int_\Omega (\nabla \cdot N_i) \hat{N}_j \, d\Omega$$

$$f = (w^h, u^{nh}) + \frac{\Delta t}{2}\Big[(w^h, f^{n+1h}) + (w^h, f^{hn}) - c(u^{nh}; w^h, u^{nh}) - a(w^h, u^{nh}) - (w, \nabla u^{nh})\Big]$$

$$- \frac{\Delta t}{2}\Big[a(w^h, u_D^{n+1}) + \sigma(w^h, u_D^{n+1}) + c(u_D^{n+1}, u^h, u_D^{n+1})\Big] - (w^h, u_D^{n+1})$$

$$h = -b(u_D^{n+1}, q^h)$$

$\left.\begin{matrix}\bar{K},\ \bar{G},\ \bar{L},\ \bar{F}_w,\ \bar{F}_q, \\ \bar{M},\ \bar{C}\end{matrix}\right\} \longrightarrow$ Matrices resulting from the introduced stabilization terms, in this case, the previously mentioned Tezduyar and Osawa stabilization.

The problem takes the form:

$$\underbrace{\begin{bmatrix} M + \bar{M} + \frac{\Delta t}{2}\left(C(u) + \bar{C} + K + \bar{K} + \sigma M\right) & \Delta t\,(G + \bar{G}) \\ -G^T + \bar{G}^T & \bar{L} \end{bmatrix}}_{A(u^{n+1})} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f^n + \bar{f}_w^n \\ \bar{f}_q^n \end{bmatrix}$$

$\hookrightarrow$ C matrix makes the problem non-linear

d) Propose an algorithm to solve the nonlinear problem

We have a problem of the type.

$$A(x^{n+1})\, x^{n+1} = b(x^n) \qquad x = \begin{bmatrix} u \\ P \end{bmatrix}$$

Given the initial conditions u_0, we will use the Picard method to solve at each time step n the following system of equations:

1. $A(x^k)\, x^{k+1} = b(x^n)$ b is known from previous time step

   Solve for x^{k+1}

2. update $A(x^k) = A(x^{k+1})$ $k = k+1$

3. Repeat until convergence

4. Advance to next time step

   $x^{n+1} = x^{k+1}$

   $n = n+1$

5. Repeat for all time steps.

e) $\sigma = 0$, flow around an object

For the simulation with $Re = 100$, both methods converge as expected, since NR presents quadratic convergence while Picard presents linear convergence (slower but more robust method)

As Re grows, convection effects become more significant and instabilities are introduced, as we see for the simulation with $Re = 1000$, for which the Newton-Raphson method diverges. Picard method has a larger radius of convergence and is therefore more stable. However, if we continue to increase the Reynolds number of the flow, at some point the Picard method will fail to reach convergence as well.

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