Omar Taha

Cauchy momentum equation

Show that $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} - \nabla\cdot\bar{\bar{\sigma}} = \rho\vec{b}$ (1) is equivalent to

$$\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\otimes\vec{v} - \bar{\bar{\sigma}}) = \rho\vec{b} \quad (2)$$

equation (2) can be written as: $\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) \underbrace{- \nabla\cdot\bar{\bar{\sigma}}} = \underbrace{\rho\vec{b}}$

those terms are the same in both equations

we only need to prove that:

$$\boxed{\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = \frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) \quad \text{equation (3)}}$$

$$\underbrace{\rho\frac{\partial \vec{v}}{\partial t}} + \rho(\vec{v}\cdot\nabla)\vec{v} = \underbrace{\rho\frac{\partial \vec{v}}{\partial t}} + \vec{v}\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}\otimes\vec{v})$$

working with index notation:

$$\rho\left(v_j\frac{\partial}{\partial x_j}\right)v_i = v_i\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j)$$

$$\rho v_j\frac{\partial v_i}{\partial x_j} = v_i\frac{\partial \rho}{\partial t} + v_i v_j\frac{\partial \rho}{\partial x_j} + \rho\frac{\partial (v_i v_j)}{\partial x_j}$$

$$\underbrace{\rho v_j\frac{\partial v_i}{\partial x_j}} = v_i\frac{\partial \rho}{\partial t} + v_i v_j\frac{\partial \rho}{\partial x_j} + \rho v_i\frac{\partial v_j}{\partial x_j} + \underbrace{\rho v_j\frac{\partial v_i}{\partial x_j}}$$

$$0 = v_i\left(\frac{\partial \rho}{\partial t} + v_j\frac{\partial \rho}{\partial x_j} + \rho\frac{\partial v_j}{\partial x_j}\right)$$

Scanned by CamScanner

$$\boxed{0 = v_i \left(\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} \right)}$$

So far we have this expression. Recalling continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0$$

Writting that continuity equation in index notation:

$$\underbrace{\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j}}_{\text{this term is equal to the term in the previous equation}} = 0$$

going back to the the equation we were trying to show:

$$0 = v_i \underbrace{\left(\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} \right)}_{0}$$

$$0 = v_i \cdot 0 \longrightarrow \boxed{0 = 0 \text{ for any component}}$$

then we showed that equation (3) the terms on the L.H.S are equal to the terms on the R.H.S

and with that we showed that equation (1) is equivalent to eq (2)