Universitat Politècnica de Catalunya - Master of science in Computational Mechanics

FINITE ELEMENTS IN FLUIDS

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Assignment 3

$$\begin{cases} u_t + uu_x = \varepsilon u_{xx}, & \text{for } (x,t) \in [-1,1] \times [0,T] \\ u(x,0) = u_0(x), & \text{for } x \in [-1,1] \; \Omega \\ u(-1,t) = u(1,t), & \text{for } t \in [0,T]. \end{cases}$$

write the one-step and two-step Taylor-Galerkin methods for the perturbed Burgers' equation

a) One-step Taylor-Galerkin:

the second-order Taylor expansion of time states $u^{n+1} = u^n + \Delta t\, u_t^n + \frac{\Delta t^2}{2} u_{tt}^n$

but from the perturbed burgers' equation $u_t = -\underbrace{uu_x}_{f_x} + \varepsilon u_{xx}$ $\left(f = \frac{u^2}{2}\right)$, thus

$$u_t = -f_x + \varepsilon u_{xx} \Rightarrow u_{tt} = -f_{xt} + \varepsilon u_{xxt} = -f_{tx} + \varepsilon u_{txx}$$

but from the chain rule we know $f_t = f_u u_t$ and substituting $u_t$ again we get

$$-(f_u u_t)_x + \varepsilon(-f_x + \varepsilon u_{xx})_{xx} = -(f_u u_t)_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx}) \Rightarrow$$

$$\Rightarrow \quad u_{tt} = -\left[f_u(-f_x + \varepsilon u_{xx})\right]_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx}) \Rightarrow$$

$$\Rightarrow \quad u_{tt} = -(-uf_x + \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})$$

thus, substituting $u_t$ and $u_{tt}$ on the Taylor expansion and rearranging yields

$$\frac{u^{n+1} - u^n}{\Delta t} = -f_x + \varepsilon u_{xx} + \frac{\Delta t}{2}\left[-(-uf_x + \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})\right]$$

multiplying by the test function $w$ and integrating over the domain

$$\int_\Omega w \frac{u^{n+1} - u^n}{\Delta t} dx = \int_\Omega w(-f_x + \varepsilon u_{xx}) dx + \frac{\Delta t}{2} \int_\Omega w\left[(uf_x - \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})\right] dx$$

integrating the terms on the r.h.s by parts yields:

$$\int_\Omega w \frac{u^{n+1}-u^n}{\Delta t} dx = -wf\Big|_\Omega + \int_\Omega w_x f \, dx + \varepsilon w u_x\Big|_\Omega - \int_\Omega \varepsilon u_x w_x \, dx +$$

$$+ \frac{\Delta t}{2}\Big\{ wu(f_x - \varepsilon u_{xx})\Big|_\Omega - \int_\Omega w_x u (f_x - \varepsilon u_{xx}) \, dx +$$

$$+ \varepsilon w(-f_{xx} + \varepsilon u_{xxx})\Big|_\Omega - \varepsilon \int_\Omega w_x (-f_{xx} + \varepsilon u_{xxx}) \, dx$$

$$\int_{-1}^{1} w \frac{u^{n+1}-u^n}{\Delta t} dx = \int_{-1}^{1} w_x (f - \varepsilon u_x) \, dx + \frac{\Delta t}{2}\Big\{ -\int_{-1}^{1} w_x \big[u(f_x - \varepsilon u_{xx}) - \varepsilon(f_{xx} - \varepsilon u_{xxx})\big] dx \Big\} +$$

$$+ w(-f + \varepsilon u_x)\Big|_{-1}^{1} + \frac{\Delta t}{2} w\big[u(f - \varepsilon u_{xx}) - \varepsilon(f_{xx} - \varepsilon u_{xxx})\big]_{-1}^{1}$$

b) Two-step Taylor-Galerkin

a two step scheme is given by : $\begin{cases} u^{n+1/2} = u^n + \frac{\Delta t}{2} u_t^n & (1) \\ u^{n+1} = u^n + \Delta t \, u_t^{n+1/2} & (2) \end{cases}$

substituting $u_t = -f_x + \varepsilon u_{xx}$ on (2) yields $\frac{u^{n+1}-u^n}{\Delta t} = -(f_x + \varepsilon u_{xx})^{n+1/2}$

mutiplying by $w$ and integrading :

$$\int_{-1}^{1} w \frac{u^{n+1}-u^n}{\Delta t} dx = \int_{-1}^{1} w(-f_x + \varepsilon u_{xx})^{n+1/2} dx = w(-f + \varepsilon u_x)^{n+1/2}\Big|_{-1}^{1} - \int_{-1}^{1} w_x (-f + \varepsilon u_x)^{n+1/2} dx$$

the values on the integrals with the $(n+1/2)$ superscript are evaluated at the intermediary point given by equation (1)